TRAVELING WAVES ON ELECTRIC POWER SYSTEMS

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The problems of traveling waves on the transmission lines of a power system[1] differ considerably from those of traveling waves on telephone or telegraph circuits. The primary object in the case of the former is to know how to protect the system from abnormal voltage disturbances which might damage apparatus or cause discontinuity of service; whereas the object in the case of the latter is the transmission of signals. Attenuation, distortion, wave shape modification, and successive reflections are deliberately sought after on the power system as a means of rendering the surges innocuous, but these effects must be carefully avoided or nullified on the communication circuits so as to preserve the wave shape and transmit the signal with strength, fidelity, and without interference. On the power lines, the surges often originate from unknown causes, or at the point of origin are of unknown magnitude and shape (except from a statistical point of view); while on the communication circuits the initial shape and magnitude of the wave train are known with exactness. External fields (due to charged clouds), corona, flashovers, faults, and so on are of great importance with respect to surges; but are of no concern in the normal functioning of a telephone or telegraph line. Thus on the power lines surges originate from external or undesirable causes and every effort must be made to withstand or control them; while on communication circuits the transients are the direct means to the end.

These differences have led to corresponding differences in the mathematical approach. The power engineer is satisfied with approximations which would be intolerable to the communications engineer, and he is willing to take a license with mathematical rigor which would make any self-respecting mathematician groan. Higher mathematics has found little or no application in the study of surges on power systems. This has been due primarily to the fact that the boundary conditions are not definite enough to justify purely mathematical refinements; particularly since engineering results must be obtained in a short time by men who are not mathematicians. However, there are numerous aspects of the problem which lend themselves to mathematical excursions. It is my purpose in this lecture to

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[1] *Traveling Waves on Transmission Systems*, L. V. Bewley, New York, Wiley.

give you the engineering methods, and perhaps, here and there, to suggest how advanced mathematics might be applied.

The general problem. Figure 1 indicates a multi-conductor transmission line, protected by a ground wire[2] at the peak of the towers, terminating at a power station containing a lightning arrester, a circuit breaker, a transformer, and a generator. A cloud, floating over the line, has been charged by the action of the rising air currents on

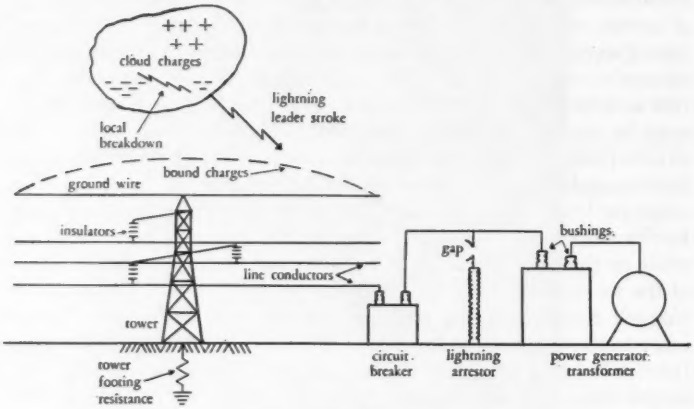

FIG. 1

the falling raindrops, and this cloud charge has induced a stationary counter-charge on the conductors of the transmission line. As the potential of the cloud increases, local breakdowns occur throughout its mass, thereby uniting some of its regions and making available, through partially ionized paths, a reservoir of charge. Eventually the field gradient reaches an intensity sufficient to initiate a leader stroke or dart, which starts towards ground. The progress of this dart is not continuous, but by jerks, each jerk depending upon the supply of additional charge to the head of the dart. It is like the armored force in a blitzkrieg break-through—it strikes to the limit of its capabilities and must then wait for tactical reinforcements and logistical support before renewing its attack. As the dart approaches earth, the field

[2] The purpose of a ground wire is to intercept a lightning stroke and thereby protect the power conductors from destructive surges. Those surges which do come in to the station are absorbed by the lightning arrestor.

gradient at the transmission line increases and this causes a migration of charge up through the towers onto the ground wire, and from the remote parts on the line conductors towards the region of field concentration. The dart finally makes contact—say with the ground wire at the tower—and a lightning surge moves out in both directions on the ground wire, inducing waves on the line conductors. But when these waves reach the next tower, reflections occur, and very soon all the neighboring spans are filled by numerous waves reflecting back and forth, and perhaps flashovers have taken place to the line conductors. These waves are rushing, with the speed of light, towards the power station, where they may enter the windings of transformers and generators, causing steep gradients which may breakdown the turn-to-turn insulation, and oscillations which may develop destructive voltages on the major insulation to ground. Perhaps a bushing flashover, or an insulation failure will cause the circuit-breaker to function, interrupting the normal 60 cycle power current, and this operation will initiate a new transient which is called a "switching surge." And perhaps an insulator flashover out on the line will culminate in intermittent arcing which may result in a cumulative building-up of dangerous voltages—the so-called "arcing ground."

The engineer does not tackle this problem in its entirety by attempting to include all the terminal apparatus as boundary conditions. Rather is he compelled to make a piecemeal attack, by handling each part of the problem as a separate and independent proposition, thereby isolating and defeating them in detail. To this end, certain aspects of the problem will be considered under four main headings.

Multi-velocity waves (tensor notation). Consider a system of n overhead transmission line conductors with voltages e_r currents i^r, charges Q^r, and fluxes ϕ_r. Then in terms of Maxwell's electrostatic potential coefficients p_{rs}

$$e_r = p_{rs}Q^s \tag{1}$$

from which

$$Q^r = K^{rs}e_s \tag{2}$$

where K^{rs} is the inverse of p_{rs} in the matrix $[p_{rs}]$. These coefficients are calculated for parallel cylindrical conductors in the presence of ground by including the images of the conductors in the ground surface.

The magnetic flux linkages are given in terms of the inductance coefficients by

$$\phi_r = L_{rs} i^s. \tag{3}$$

There are leakage currents flowing to ground and between conductors of amount $G^{rs} e_s$. The G^{rs} coefficients are supposed to include the effects of both leakage and corona.

And finally, there are resistance drops $R_{rs} i^s$ in the conductors due to the flow of currents.

Herefrom, the differential equations for the multi-conductor transmission system become (putting $p = \partial/\partial t$, in the Heaviside sense)

$$-\frac{\partial e_r}{\partial x} = p\phi_r + R_{rt} i^t = (L_{rt} p + R_{rt}) i^t, \tag{4}$$

$$-\frac{\partial i^t}{\partial x} = pQ^t + G^{ts} e_s = (K^{ts} p + G^{ts}) e_s. \tag{5}$$

Eliminating i^t there results

$$\left(J_r^s - \delta_r^s \frac{\partial^2}{\partial x^2}\right) e_s = 0 \tag{6}$$

in which δ_r^s is the Kronecker delta and

$$J_r^s = (L_{rt} p + R_{rt})(K^{ts} p + G^{ts}). \tag{7}$$

Now if the losses are ignored ($R_{rs} = 0$, $G^{rs} = 0$), (6) is satisfied by the traveling wave[3]

$$e_s = a_s^{(\alpha)} f_{(\alpha)}(x - v_{(\alpha)} t) \tag{8}$$

which substituted in (6) gives

$$\left(L_{rt} K^{ts} p^2 - \delta_r^s \frac{\partial^2}{\partial x^2}\right) a_s^{(\alpha)} f_{(\alpha)}(x - v_{(\alpha)} t) = 0. \tag{9}$$

Since this equation must be satisfied for waves of the same velocity

$$(L_{rt} K^{ts} v_{(\alpha)}^2 - \delta_r^s) a_s^{(\alpha)} = 0. \tag{10}$$

The velocities are given by the roots of the determinant

[3] An enclosed index is used here to suspend the summation convention.

$$(11) \qquad \left| L_{rt} K^{ts} v_{(\alpha)} - \delta^s_r \right| = \left| c^s_r \right| = 0.$$

To each root of (11) there correspond n values of $a^{(\alpha)}_s$, and any $(n-1)$ of them may be determined in terms of one value taken arbitrarily. Let this one value be $a^{(\alpha)}_1 = 1$. Solving (10)

$$(12) \qquad a^{(\alpha)}_s = \frac{-A^r_s}{\left| b^s_r \right|} (L_{rt} K^{t1} v_{(\alpha)}) \quad \text{where } s \neq 1, r \neq 1,$$

in which $\left| b^s_r \right|$ is $\left| c^s_r \right|$ with the $s=1$ row and $r=1$ column deleted, and A^r_s is the cofactor of b^s_r in $\left| b^s_r \right|$.

The complete solution then becomes

$$(13) \qquad e_r = a^\alpha_r [f_\alpha(x - v_{(\alpha)}t) + F_\alpha(x + v_{(\alpha)}t)].$$

By (5) the corresponding currents are

$$(14) \qquad i^r = -p \int K^{rs} e_s dx = K^{rs} a^\alpha_s (f_\alpha - F_\alpha) v_{(\alpha)} = Y^{r\alpha} (f_\alpha - F_\alpha).$$

Transition points. Consider the general case of Figure 2 in which any number of incoming lines terminate at a transition point consisting of an inter-connected network and any number of outgoing

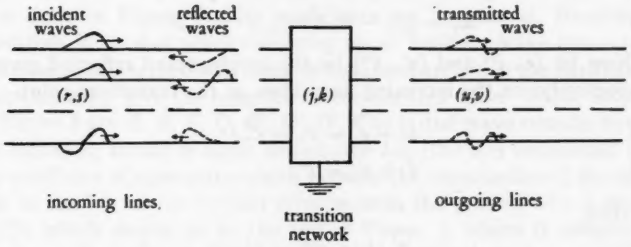

FIG. 2

lines. When the incident waves on the incoming lines reach the transition point, currents will flow into the network, transmitted waves will move out on the outgoing lines, and reflected waves will start back on the incoming lines. In tensor notation let:

z_{rs} = surge impedances of incoming lines,[4]

[4] The surge impedance of a line is the coefficient of proportionality between its voltage and current ($e = zi$). Its reciprocal is called surge admittance. For surges on transmission lines, these parameters are essentially constant, and are an indication of the current associated with a surge of given voltage.

y^{rs} = surge admittances of incoming lines (inverse of z^{rs}),
z_{uv} = surge impedances of outgoing lines,
y^{uv} = surge impedances of outgoing lines (inverse of z_{uv}),
Z_{jk} = branch impedances of the network,
$C^{\alpha}_{\alpha'}$ = transformation tensor specifying the total interconnections of the network and outgoing lines.

Then

(1) $Z_{\alpha\beta} = (Z_{jk} + z_{uv})$ = impedance of network and outgoing lines before interconnection,

(2) $Z_{\alpha'\beta'} = C^{\alpha}_{\alpha'} C^{\beta}_{\beta'} Z_{\alpha\beta}$ = impedance after interconnection of the network and outgoing lines.

Now $Z_{\alpha'\beta'}$ may include branches other than those connected to the incoming lines. The open circuit branches will have been eliminated by $C^{\alpha}_{\alpha'}$, but the branches other than those connected to the incoming lines will have to be eliminated by the substitutions:

$$Z_{\alpha'\beta'} = Z_{rs} + Z_{rq} + Z_{ps} + Z_{pq}, \tag{3}$$

$$E_r = Z_{rs} I^s + Z_{rq} I^q, \tag{4}$$

$$0 = Z_{ps} I^s + Z_{pq} I^q, \tag{5}$$

from which

$$E_r = (Z_{rs} - Z_{rq} Y^{qv} Z_{ps}) I^s = Z'_{rs} I^s . \tag{6}$$

Now let (e_r, i^r) and (e'_r, i'^r) be the incident and reflected waves, respectively, on the incoming line. Then at the transition point

$$e_r + e'_r = E_r = Z'_{rs} I^s, \tag{7}$$

$$i^r + i'^r = I^r, \tag{8}$$

so that

$$e_r + e'_r = Z'_{rs}(i^r + i'^r) = Z'_{rs} y^{st}(e_t - e'_t) \tag{9}$$

from which

$$(\delta^t_r + Z'_{rs} y^{st}) e'_t = (-\delta^t_r + Z'_{rs} y^{st}) e_t. \tag{10}$$

This system of equations defines the reflected voltage waves e'_r.

The total voltage at the transition point then follows by (7), the total current by (8), the remaining network currents by (5) and the network voltages by (4), and so on.

Successive reflections. The calculation of successive reflections is oftentimes a long and involved process; particularly in those cases

where reflections may occur from a whole series of neighboring junctions. An example is the case of lightning striking the ground wire at midspan. The incident waves move out in opposite directions until they reach the nearest towers, where they reflect as a consequence of the surge impedance of the continuing ground wire in parallel with that of the tower. Transmitted waves quickly reach the foot of the tower from which they reflect as a result of the ground resistance. Other waves reflect from the next tower, and from the next after that, and so on. Thus within a few microseconds the system is alive with a whole series of waves moving in different directions, arriving at different times, of different magnitudes and polarity, and having experienced different attenuations and distortions.

In order to keep track of all these components a *lattice diagram* has been devised, such as shown in Figure 3 for the case of lightning striking a ground wire at midspan. The progress of each wave component is easily followed as it slides downhill along its zig-zag path, giving rise to reflections at each junction. Thus at any instant of time the waves at all points on the line can be identified; or at any point on the line the time of arrival of each wave can be seen.

To construct such a reflection lattice it is first necessary to determined the reflection and refraction coefficients[5] at each junction, and to post these on the sketch of the system being studied, as has been done on Figure 3. The coefficients are, in general, Heaviside operators, such that when operating on an incident wave (regarded as a time function reckoned from its instant of arrival at the particular junction) they give the reflected or transmitted wave. Those shown in Figure 3 are A, B, C, D, B', C', D'. The initial wave coming down the lightning stroke of surge impedance $2Z_0$ (the 2 is occasioned by the condition of symmetry which permits the amputation of the system to the left of the stroke) refracts onto the ground wire a wave $A \cdot f(t)$ which moves on to the top of Tower 1, where it reflects a portion $AB' \cdot f(t)$ back towards its origin, and transmits a portion $AB \cdot f(t)$ to the next ground wire section and also down the tower. When the wave reaches the foot of the tower it reflects a portion $ABD' \cdot f(t)$ back up the tower. Likewise reflections return from the top and foot of Tower 2 and from the towers beyond. Now each of these reflections could be traced out independently on the lattice and all waves fully accounted for. But the labor is great. The work can be

[5] A "reflection coefficient" is the operator determined from equation (10) of the previous section on transition points, which permits the calculation of a reflected wave in terms of the incident wave. Likewise, the "refraction coefficient" gives the transmitted wave.

simplified by introducing the concept of "wave trains," and "retarder operators."

The system of waves reflected back on the ground wire due to the arrival at a tower top of a wave $f(t')$ and the successive reflections

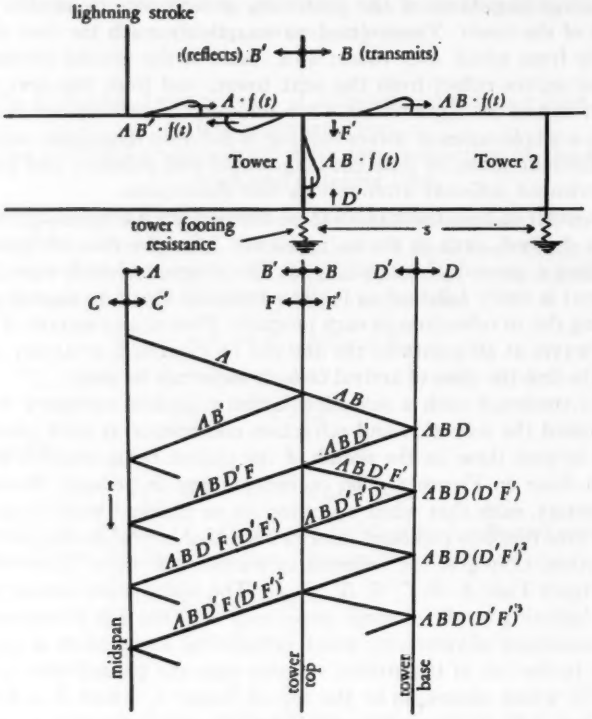

FIG. 3

up and down the tower is seen from the lattice to be a "wave train" of *Type* I:

$$
\begin{aligned}
e' &= B' \cdot f(t') + BD'F \cdot f(t' - 2h) + BD'F(D'F') \cdot f(t' - 4h) + \cdots \\
&= B'\left[f(t') + \frac{BD'F}{B'} \sum (D'F')^{m-1} f(t' - 2hm) \right] \qquad (1) \\
&= B'[1 + a \sum b^{m-1} \cdot \phi(m)] f(t') \\
&= B' \alpha_m f(t)
\end{aligned}
$$

in which $\phi(m)$ is a "retarder operator" such that the time of arrival of a wave to which it is applied is retarded by $(2hm)$, thus

$$\phi(m)\cdot f(t') = f(t' - 2hm). \tag{2}$$

The wave train transmitted on to the next tower by the incident wave and its tower reflection is the *Type* II wave train

$$\begin{aligned} e'' = e + e' &= B\left[f(t') + \frac{D'F}{B}\sum (D'F')^{m-1}\cdot f(t' - 2hm)\right] \\ &= B[1 + c\sum b^{m-1}\phi(m)]f(t') \\ &= B\beta_m f(t'). \end{aligned} \tag{3}$$

In terms of the wave trains of Types I and II the complete history of the reflections can now be written down.

The initial wave transmitted by a lightning surge to midspan is $Af(t)$ which arrives at the first tower at time $0.5s$ and gives rise to the Type I wave train of the first order.

$$e_1 = AB'\alpha_m f(t - 0.5s). \tag{4}$$

This wave train arrives at midspan at time s and reflects therefrom as

$$e_2 = AB'C'\alpha_m f(t - s). \tag{5}$$

When the reflected wave train arrives at the tower at time $(1.5s)$ it generates a new Type I wave train of the second order

$$e_3 = AB'C'\alpha_m\alpha_n f(t - 1.5s). \tag{6}$$

Continuing this process we find combinations of the form $\alpha_m\alpha_n\alpha_p\cdots$ in which products are to be interpreted as

$$\begin{aligned} \alpha_m\alpha_n\alpha_p &= [1 + a\sum b^{m-1}\phi(m)][1 + a\sum b^{n-1}\phi(n)][1 + a\sum b^{p-1}\phi(p)] \\ &= [1 + 3a\sum b^{m-1}\phi(m) + 3a^2\sum\sum b^{m+n-2}\phi(m+n) \\ &\quad + a^3\sum\sum\sum b^{m+n+p-3}\phi(m+n+p)]. \end{aligned} \tag{7}$$

Now in addition to those wave trains operating between midspan and Tower 1, contributions eventually arrive from neighboring towers. Thus at time $(0.5s)$ there is transmitted beyond Tower 1, owing to the initial wave $Af(t)$, the Type II wave train of the first order

$$E_1 = AB\beta_m f(t - 0.5s). \tag{8}$$

This wave train reaches Tower 2 at time $(1.5s)$ where it generates a Type I wave train of the second order

$$E_2 = ABB'\beta_m\alpha_n f(t - 1.5s). \tag{9}$$

This wave train arrives at Tower 1 at time $(2.5s)$ and generates a third order wave train of Type I

$$E_3 = AB^2B'\beta_m\alpha_n\beta_p f(t - 2.5s), \tag{10}$$

and this wave train, reaching midspan at $(3s)$, reflects therefrom as

$$E_4 = AB^2B'C'\beta_m\alpha_n\beta_p f(t - 3s). \tag{11}$$

With the assistance of the lattice diagram and retarder operators, the potential at any point can be written down. For example, at the top of Tower 1

$$\begin{aligned} V = {} & AB\beta_m f(t - 0.5s) + ABB'C'\alpha_m\beta_n f(t - 1.5s) \\ & + AB(BC)^2\alpha_m\alpha_n\beta_p f(t - 2.5s) + \cdots \\ & + AB^2B'\beta_m\alpha_n\beta_p f(t - 2.5s) + \cdots \\ & + \cdots . \end{aligned} \tag{12}$$

Induced lightning surges. Suppose a cloud bearing a charge Q_0 is over a transmission line, Figure 4, and is discharging either to

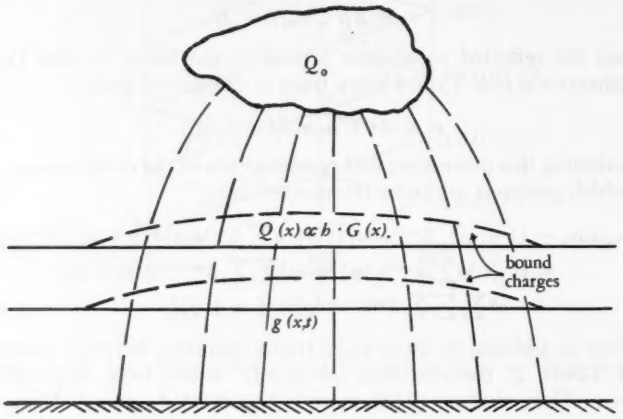

FIG. 4

ground or to another cloud according to some function of time $\psi(t)$; that is, the charge remaining in the cloud at any instant is $Q_0[1 - \psi(t)]$.

Depending upon the shape and size of the cloud charge, its height

above ground, and its position with respect to the transmission line, the line will experience a gradient

$$g(x, t) = G(x) \cdot [1 - \psi(t)] \tag{1}$$

in which $G(x)$ represents the initial distribution of gradient (at the beginning of cloud discharge) as a function of distance along the line. Under the influence of this field, charges of opposite sign to that of the cloud will leak over the insulators, or migrate from the remote parts of the system, and collect on the line conductors as bound charges. The density of bound charge at any point x will be proportional to the gradient and to the height h of the conductor above ground (since the field is substantially uniform for a hundred feet or so above ground). These bound charges nullify the potential due to the external field, so that initially the line charges are given by

$$V_r = 0 = G(x) \cdot h_r + p_{rs} Q^s, \qquad r = 1, 2, \cdots, n. \tag{2}$$

Suppose then m of the n wires are ground wires perfectly grounded throughout their length, and let these ideal ground wires be represented by (j, k) indices. The remaining $(n-m)$ wires are power conductors, and will be represented by (u, v) indices.

Now if the field gradient $G(x)$ is suddenly removed, the bound charges on the line wires will not change at the first instant, but those on the ground wires are instantly replaced by new charges Q' since the ground wires must remain at zero potential. Therefore

$$0 = p_{jk} Q'^k + p_{jv} Q^v, \tag{3}$$

$$V_u = p_{uk} Q'^k + p_{uv} Q^v. \tag{4}$$

From (2) and (3) all the charges Q^s and Q'^k may be found; and from (4) the potentials V_u may be determined. These potentials immediately move out as pairs of traveling waves (in opposite directions). At the first instant, however, the forward and backward waves $f_u(x-vt)$ and $F_u(x+vt)$ add up to the voltages given by (4), and the resultant current flow must be zero in the isolated power conductors. Thus

$$f_u + F_u = V_u, \tag{5}$$

$$Y^{uv}(f_v - F_v) = 0. \tag{6}$$

Hence

$$F_u = f_u = V_u/2, \tag{7}$$

that is, the forward and backward waves on a conductor are the same shape and magnitude.

However, the release of the bound charge is not instantaneous, but according to the law of cloud discharge $\psi(t)$. The corresponding traveling waves are then given by Duhamel's theorem

$$(8)\quad e = \int_0^t \{f[x + v(t - \tau)] + f[x - v(t - \tau)]\} \frac{\partial \psi(\tau)}{\partial \tau} d\tau$$

$$(9)\quad = \lim_{\Delta t \to 0} \sum_0^n \{f[x + v(n - k)\Delta t] + f[x - v(n - k)\Delta t]\} \Delta \psi_k$$

where $n \cdot \Delta t = t$, $k \cdot \Delta t = \tau$ and $\Delta \psi_k = \psi[(k+1)\Delta t] - \psi[k \cdot \Delta t]$. The application of the integral is limited to relatively simple expressions for f and ψ, but the summation can be used for any functions whose graphs are known or assumed. Ultimately, since both f and ψ derive from experimental data, it is best to use the summation expression. Both graphical and tabular methods have been devised for its ready application, and engineering solutions are quickly arrived at.

Equation (8) can also be derived by setting up the conditions in terms of retarded potentials. Very few engineers deal with retarded potentials, whereas a considerable number of them are familiar with Duhamel's theorem through Heaviside's operational calculus.

LEHIGH UNIVERSITY

SOME NEW METHODS OF SOLUTION OF TWO-DIMENSIONAL PROBLEMS IN ELASTICITY

IVAN S. SOKOLNIKOFF

1. **Introduction.** The main purpose of this address is to bring to the attention of the workers in the theory of elasticity and related branches of applied mathematics a simple general method of solution of several important classes of two-dimensional boundary value problems. I use the term "two-dimensional" or "plane" boundary value problems in the sense that their mathematical formulation requires the introduction of only two independent variables. In this sense the problems of St. Venant on torsion and flexure of cylinders, and the problems on deflection and buckling of elastic plates, which have a three-dimensional physical aspect, are two-dimensional.

The method which I intend to discuss was developed mainly by a group of Russian mathematicians, and despite the fact that it has been utilized extensively in Russia for more than a decade, it is virtually unknown in this country. A great variety of problems to which it has been applied to obtain useful solutions includes an investigation of flexure and torsion of beams, a study of thermo-elastic stresses in composite cylinders, an analysis of deflection of anisotropic plates, and a multitude of problems characterized by the states of plane stress and plane strain.

Inasmuch as familiarity with the concepts of applied mathematics is a rare virtue, I shall reduce the use of the technical language to a minimum, and shall ask you to take for granted certain basic equations of the theory of elasticity. Failure to comprehend the origin of these equations will not impair the understanding of the general method of their solution.

We shall suppose that a two-dimensional region R, occupied by an elastic medium, is referred to a system of cartesian axes (x, y). To fix the ideas we can think of the region R as representing the cross-section of a long cylinder whose elements are parallel to the z-axis, and whose lateral surface is subjected to a distribution of external forces that is independent of the z-coordinate. Under the action of such forces, the medium, in general, will be distorted, and the displacement of the points of the region R in the directions of the x- and y-axes will be denoted by $u(x, y)$ and $v(x, y)$, respectively. If the medium

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is assumed to be isotropic, then the stresses produced in the medium are connected with the derivatives of u and v by the formulas (Hooke's law)

$$(1.1)\quad \tau_{xx} = \lambda\Delta + 2\mu\frac{\partial u}{\partial x}, \quad \tau_{yy} = \lambda\Delta + 2\mu\frac{\partial u}{\partial y}, \quad \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

where the τ's are the components of the stress tensor, λ and μ are elastic constants introduced by Lamé, and $\Delta \equiv \partial u/\partial x + \partial v/\partial y$.

If the medium is in equilibrium (in the absence of body forces) then the components of the stress tensor τ satisfy throughout the interior of the region the differential equations

$$(1.2)\qquad \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0,$$

while on the boundary C of the region R they fulfill the conditions

$$(1.3)\qquad \begin{aligned} \tau_{xx}\cos(x, v) + \tau_{xy}\cos(y, v) &= X(s) \\ \tau_{xy}\cos(x, v) + \tau_{yy}\cos(y, v) &= Y(s), \end{aligned}$$

where v denotes the exterior normal to the contour C, and X and Y are the x- and y-components of the prescribed external force estimated per unit length s of the contour C.

I shall call the problem of determining the solution of the system of equations (1.1), (1.2), (1.3), for the five unknown functions τ_{xx}, τ_{xy}, τ_{yy}, u, v, the first boundary value problem of the plane theory of elasticity. Instead of specifying the distribution of external force on the boundary C of the region R, we can impose the requirement that the displacements u and v assume prescribed values on the contour C. This latter problem will be referred to as the second boundary value problem.

The great majority of the two-dimensional boundary value problems in the theory of elasticity is reducible to the solution of these boundary value problems. We shall be concerned only with those solutions in which the components of the stress tensor are continuous and single-valued throughout the region R together with their first and second partial derivatives with respect to x and y. The displacements u an v will be assumed to be continuous and single-valued in the region R together with their partial derivatives including those of the third order. These restrictions arise essentially from physical considerations, and it is a consequence of the restrictions imposed on the displacements u and v that the components of the

stress tensor determined from equations (1.2) satisfy in the region R the Beltrami-Michell equation of compatibility, namely

$$\nabla^2(\tau_{xx} + \tau_{yy}) = 0, \tag{1.4}$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$.

It is clear that equations (1.2) represent a necessary and sufficient condition for the existence of a function $U(x, y)$, such that

$$\tau_{xx} = \frac{\partial^2 U}{\partial y^2}, \qquad \tau_{yy} = \frac{\partial^2 U}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \tag{1.5}$$

and it follows at once from (1.4) that $U(x, y)$ satisfies the biharmonic equation

$$\nabla^2 \nabla^2 U = 0 \tag{1.6}$$

throughout the region R. The function U is commonly known as Airy's stress function.

Substituting the relations (1.5) in the boundary conditions (1.3) and combining results lead to a compact boundary condition in the form

$$\frac{d}{ds}\left(\frac{\partial U}{\partial y} - i\frac{\partial U}{\partial x}\right) = X + iY, \qquad \text{on } C, \tag{1.7}$$

where d/ds represents differentiation along the arc length of the contour C and $i^2 = -1$. Inasmuch as the right-hand member of (1.7) is a known function of the points of the contour C, it is clear that the solution of the first boundary value problem is reduced to the determination of the biharmonic function $U(x, y)$ whose derivatives, $\partial U/\partial x$ and $\partial U/\partial y$, are specified functions of the points of C. This latter problem was subject to a prize offered by the Paris Academy, and its complete theoretical solution for the case of a finite simply-connected region R, bounded by a contour C satisfying certain general conditions, was obtained by J. Hadamard and G. Lauricella.[1] They solved the problem by reducing it to the solution of a Fredholm's integral equation.

A significant step in the direction of the development of the theory was made by G. V. Kolossoff, whose Dorpat dissertation[2] (1909) was concerned with the application of the theory of functions of a complex variable to the solution of plane problems of the theory of elasticity. Further important results were obtained by N. I. Mus-

[1] Among the recipients of the prize were A. Korn, J. Hadamard, and G. Lauricella. See references [1], [2], and [3] at the end of this paper.

[2] See [4–8].

chelišvili, who made use of conformal mapping and of integrals analogous to those of Cauchy to obtain an elegant solution of the first and second boundary value problems for an arbitrary simply-connected domain (finite or infinite).

In a series of papers dating from 1919, Muschelišvili gave a detailed analysis of the character of solution in simply- and multiply-connected domains and illustrated his method of solution by providing practically useful solutions of a number of specific problems [9–21].

The questions pertaining to the existence of solution in multiply-connected domains were dealt with extensively by S. G. Michlin, who gave, in the period from 1933 to 1935, several methods of theoretical solution of the first and second boundary value problems [22–26].

In the following section I shall limit myself to a discussion of Muschelišvili's method of solution of the boundary value problems of plane elasticity for simply-connected domains. While the application of this method to multiply-connected domains is important from a technical point of view (see §3) the essence of the method is adequately illustrated by a consideration of the simpler case.

2. **Reduction of the problem to a functional equation.** It is clear from equation (1.6) that the function $\nabla^2 U = P(x, y)$ is harmonic in R. Let $Q(x, y)$ denote the conjugate harmonic function, and introduce an analytic function $\phi(z)$ of a complex variable $z = x + iy$ defined by the formula

$$P + iQ = 4\phi'(z),$$

where prime denotes the derivative with respect to z. If the real and imaginary parts of the function $\phi(z)$ are denoted by p and q, respectively, then it is easily verified that

$$\nabla^2(U - px - qy) = 0.$$

Accordingly, the function

$$p_1 = U - px - qy$$

can be considered as the real part of some analytic function $\chi(z)$, and since

$$2(px + qy) \equiv \bar{z}\,\phi(z) + z\,\overline{\phi(z)},$$

one can write the biharmonic function U in the form[3]

[3] The possibility of representing a biharmonic function with the aid of two analytic functions of a complex variable was noted first by É. Goursat, Bulletin de la

$$2U = \bar{z}\,\phi(z) + z\,\overline{\phi(z)} + \chi(z) + \overline{\chi(z)}, \tag{2.1}$$

where bars denote the conjugate complex values.

Inserting the value of U given by (2.1) in the left-hand member of equation (1.7) yields the boundary condition in the form

$$(X + iY)ds = -id[\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)}], \tag{2.2}$$

where $\psi(z) = \chi'(z)$.

If both members of equation (2.2) are integrated along the contour C from some arbitrary point $s = s_0$, there results

$$\int_{s_0}^{s} (X + iY)ds = -i[\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)}] + \text{const.},$$

so that the boundary condition can be written as

$$\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)} = f_1(s) + if_2(s), \qquad \text{on } C, \tag{2.3}$$

where

$$i\int_{s_0}^{s} (X + iY)ds = f_1(s) + if_2(s).$$

The constant of integration has been omitted in the formula (2.3) since it can be fixed in an arbitrary manner without affecting the state of stress.[4]

A reference to the formulas (1.5) and (2.1) shows that the components of the stress tensor τ are easily calculable once the functions $\phi(z)$ and $\psi(z)$ are known, and it is readily checked by integrating equations (1.1) that

$$2\mu(u + iv) = -\left(\frac{\partial U}{\partial x} + i\,\frac{\partial U}{\partial y}\right) + \frac{2(\lambda + 2\mu)}{\lambda + \mu}\,\phi(z). \tag{2.4}$$

Thus the complete solution of the first boundary value problem is made to depend on the solution of the functional equation (2.3).

The formulation of the second boundary value problem in the form of a functional equation likewise presents no difficulty. In fact, the

Société Mathématique de France, vol. 26 (1898), p. 236. The derivation of Goursat's formula given above is due to N. I. Muschelišvili, Bulletin de l'Académie des Sciences de l' URSS, 1919, pp. 663–686. The analytic character of solutions of a biharmonic equation follows from this derivation, while the derivation of Goursat depends on the assumption that every biharmonic function is analytic.

[4] This statement follows from the fact that the state of stress, in a simply-connected domain, is not altered when the function $\phi(z)$ is replaced by $\phi(z)+ci\bar{z}+\alpha+i\beta$ and $\psi(z)$ by $\psi(z)+\alpha'+i\beta'$, where c, α, β, α', and β' are arbitrary real constants. See, for example, reference [15] listed at the end of this paper.

desired functional equation, in this case, follows almost at once from equation (2.4).

The limitations of time do not permit me to enter upon the discussion of the interesting questions connected with the existence and uniqueness of solution of the problem, and I shall confine myself to a remark that the proofs of uniqueness and existence of solution present no serious difficulties if one assumes the functions $\phi(z)$, $\phi'(z)$, and $\psi(z)$ to be continuous in the closed region R, the prescribed functions $X(s)$ and $Y(s)$ to satisfy Hölder's condition, and the contour C to have a continuously turning tanget.[5] It may be remarked that these restrictions are dictated by the behavior of stresses and displacement occurring normally in physical problems, and that they can be considerably relaxed by an analytically inclined mathematician.

I shall indicate in the remainder of this section some modes of attack on the problems that lead to the determination of the functions $\phi(z)$ and $\psi(z)$ in practically useful forms.

The solution of the functional equation (2.3) is almost immediate when the region R is that bounded by a unit circle $|z| \leqq 1$. Since the functions $\phi(z)$ and $\psi(z)$ are analytic in R, one can write

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \psi(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad |z| < 1. \tag{2.5}$$

The functions X and Y, characterizing the prescribed distribution of stress on the boundary of the unit circle, can be regarded as functions of the polar angle θ defined by the relation $z = e^{i\theta}$. Then the right-hand member of (2.3) can be expressed in the form

$$f_1 + if_2 = i\int_{s_0}^{s} (X + iY)ds = \sum_{-\infty}^{\infty} c_n e^{in\theta},$$

if one assumes that the functions f_1 and f_2 are of bounded variation. The Fourier coefficients c_n being known, one can insert the foregoing expansion in the right-hand member of (2.3) and the series (2.5) in the left-hand member. A comparison of like powers of $e^{i\theta}$ on both sides of the resulting equation will yield expressions for the unknown coefficients a_n and b_n in terms of the known values c_n. The justification of the validity of the formal solution obtained in this manner presents no difficulty in most problems of practical interest, since the behavior of functions describing the specified distribution of stress on the boundary cannot be too pathological.

[5] See, for example, [26].

If the contour C bounding the region R is not a circle, one can introduce a function

$$z = \omega(\zeta)$$

that maps the region R conformally on the unit circle $|\zeta| < 1$. Let the transforms of the functions $\phi(z)$ and $\psi(z)$ be denoted, respectively, by ϕ_1 and ψ_1, so that $\phi_1(\zeta) = \phi[\omega(\zeta)]$ and $\psi_1(\zeta) = \psi[\omega(\zeta)]$. Then a simple calculation shows that the functional equation (2.3) goes over into

$$\phi_1(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\phi_1'(\sigma)} + \overline{\psi_1(\sigma)} = f_1 + if_2, \tag{2.6}$$

where $\sigma = e^{i\theta}$ represents a point on the boundary γ of the circle $|\zeta| = 1$, and $f_1 + if_2$ is the transform of the right-hand member of (2.3) and hence can be regarded as a known function of θ.

The method of solution in series indicated above can be applied to determine the analytic functions

$$\phi_1(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \qquad \psi_1(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n, \qquad |\zeta| < 1,$$

and it is not difficult to verify that the coefficients a_n and b_n can be determined explicitly in terms of the Fourier coefficients of the functions $f_1 + if_2$ and $\omega(\sigma)/\overline{\omega'(\sigma)}$, whenever the mapping function $\omega(\zeta)$ is a polynomial.[6]

If it is recalled that the mapping function for a suitably restricted simply-connected region can be approximated by a polynomial $z = \omega_n(\zeta)$ of sufficiently high degree (in the sense that the polynomial $z = \omega_n(\zeta)$ maps the region $|\zeta| < 1$ on some region R_n which can be made to approximate the region R as closely as desired) then the practical usefulness of the method of solution outlined above becomes immediately obvious.

In a number of important problems the determination of the functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ can be greatly simplified by transforming the functional equation (2.6) into an equivalent pair of integral equations. Thus, consider the equation (2.6) and the equation

$$\overline{\phi_1(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\phi_1'(\sigma) + \psi_1(\sigma) = f_1 - if_2 \tag{2.7}$$

which is obtained from (2.6) by forming the conjugate expressions. If

[6] For further particulars see [13, pp. 287–296]. A detailed exposition and some specific calculations pertaining to several problems of physical interest are contained in [30, pp. 266–311].

both members of these equations are multiplied by $(1/2\pi i)d\sigma/(\sigma-\zeta)$ and the resulting expressions integrated over the contour γ of the unit circle, one obtains

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma \frac{\phi_1(\sigma)\,d\sigma}{\sigma-\zeta}+\frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma \\
&\qquad+\frac{1}{2\pi i}\int_\gamma \frac{\overline{\psi_1(\sigma)}}{\sigma-\zeta}\,d\sigma = A(\zeta), \\
&\frac{1}{2\pi i}\int_\gamma \frac{\overline{\phi_1(\sigma)}}{\sigma-\zeta}\,d\sigma+\frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\phi_1'(\sigma)}{\sigma-\zeta}\,d\sigma \\
&\qquad+\frac{1}{2\pi i}\int_\gamma \frac{\psi_1(\sigma)}{\sigma-\zeta}\,d\sigma = B(\zeta),
\end{aligned}
\tag{2.8}
$$

where

$$A(\zeta)=\frac{1}{2\pi i}\int_\gamma[(f_1+if_2)/(\sigma-\zeta)]d\sigma,\quad B(\zeta)=\frac{1}{2\pi i}\int_\gamma[(f_1-if_2)/(\sigma-\zeta)]d\sigma.$$

The complete equivalence of the simultaneous equations (2.8) with (2.6) follows at once from the theorem of Harnack. The equations (2.8) can be simplified by applying Cauchy's integral formula, and by observing that

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{f(\sigma)}}{\sigma-\zeta}\,d\sigma=\overline{f(0)},$$

where $f(\zeta)$ is any function continuous in the closed region $|\zeta|\leqq 1$ and analytic in the interior of the unit circle γ. Making use of these simplifications leads to a pair of integral equations

$$
\begin{aligned}
&\phi_1(\zeta)+\frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma+\overline{\psi_1(0)}=A(\zeta), \\
&\psi_1(\zeta)+\frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\phi_1'(\sigma)}{\sigma-\zeta}\,d\sigma+\overline{\phi_1(0)}=B(\zeta).
\end{aligned}
\tag{2.9}
$$

It was demonstrated by Muschelišvili that the solution of these equations for the unknown functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ can be obtained in an elementary way in terms of the integrals of Cauchy's type whenever the mapping function $\omega(\zeta)$ is rational.[7]

[7] See, for example, [15, 16].

The system of integral equations (2.9) is not of the standard type, but it is possible to reduce the first of equations (2.9) to an ordinary integral equation of Fredholm, of the second kind.[8] Once the function $\phi_1(\zeta)$ is determined from this equation, the second of equations (2.9) permits one to calculate the function $\psi_1(\zeta)$ by quadratures. Some effective methods of numerical solution of such equations were given recently by E. I. Nyström, D. I. Šerman, A. Gorgidze, A. Rukhadze and others [27–29].

3. **Thermal stresses.** One of the technically important problems is that of determining the state of stress produced in an elastic body by heating. Consider a body in the shape of a right cylinder whose length is large compared with the linear dimensions of the cross-section, and let the xy-plane of the cartesian coordinate system lie in one of the cross-sections of the cylinder. If such a body is subjected to a change in temperature which is a function of the x- and y-coordinates alone, then the change of temperature will cause the body to expand. In general, the expansion of an elastic body cannot proceed freely and the body becomes stressed. The problem of determining the distribution of thermal stresses assumes particular importance in a study of compound columns. A concrete beam reenforced by steel and subjected to varying temperatures is a specific example.

It is not difficult to see that the determination of stresses in this case essentially reduces to the solution of a system of equations which differ from the system (1.1), (1.2), (1.3) only in that Hooke's law (1.1) is replaced by a more general one that takes into account the effect of thermal expansion on stresses. This generalized law has the form

$$\tau_{xx} = \lambda\Delta + 2\mu\frac{\partial u}{\partial x} - (3\lambda + 2\mu)\alpha T(x, y),$$

$$\tau_{yy} = \lambda\Delta + 2\mu\frac{\partial v}{\partial y} - (3\lambda + 2\mu)\alpha T(x, y),$$

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

[8] This reduction is accomplished essentially by differentiating both members of the equation with respect to ζ and by allowing ζ to approach σ_0, where σ_0 is an arbitrary point of the contour γ. The resulting equation has the form $\phi'(\sigma_0)+(1/2\pi i)\int_\gamma K(\sigma_0, \sigma)\overline{\phi'(\sigma)}d\sigma = F(\sigma_0)$, where $F(\sigma_0)$ is a known function and the kernel $K(\sigma_0, \sigma)$ is a continuous function provided that suitable restrictions are imposed on the mapping function $\omega(\zeta)$. See, for example, [13, 16].

where α is the coefficient of the linear thermal expansion and $T(x, y)$ is the known change in temperature of the body.

If we introduce a stress function $U(x, y)$ defined by the formulas (1.5) and perform calculations analogous to those that led to the biharmonic equation (1.6), there will result the equation[9]

$$\nabla^2\nabla^2 U + k\nabla^2 T = 0, \tag{3.1}$$

where $k = 2\mu(3\lambda+2\mu)(\lambda+2\mu)^{-1}\alpha$.

Setting $U = \Phi - V$, where V is a suitable solution of the Poisson equation

$$\nabla^2 V = kT,$$

shows that equation (3.1) is equivalent to a system of two equations

$$\nabla^2\nabla^2\Phi = 0, \quad \nabla^2 V = kT.$$

A calculation in every respect parallel to the one that led to the boundary condition (1.7) in this case yields

$$\frac{d}{ds}\left(\frac{\partial\Phi}{\partial y} - i\frac{\partial\Phi}{\partial x}\right) = X + iY + \frac{d}{ds}\left(\frac{\partial V}{\partial y} - i\frac{\partial V}{\partial x}\right), \quad \text{on } C, \tag{3.2}$$

and it follows that the thermo-elastic problem is identical with the first boundary value problem discussed in §§1 and 2.

An assertion of the mathematical equivalence of the two problems does not settle many specific questions regarding the behavior of cylinders when subjected to heat. The implications of the connection of the derivatives of the function Φ with those of the function V as expressed in formula (3.2) were analyzed by B. E. Gatewood in a doctoral dissertation.[10] Gatewood also studied the thermo-elastic problem for cylinders with longitudinal cavities and obtained some interesting results for cylinders composed of different materials. These results, however, are too specialized to be entered upon in this address.

It is obvious that a similar mode of attack can be expected to succeed in a study of thermal stresses in elastic plates made up of several layers of different materials.[11]

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[9] Of course, the equation of compatibility, in this case, assumes a form different from (1.4).

[10] *Thermal stresses in long cylindrical bodies*, dissertation, Wisconsin, 1939, 96 pp. See, also, B. E. Gatewood, this Bulletin, abstract 45-9-314. A paper based on this dissertation appears in Philosophical Magazine, (7), vol. 32 (1941), pp. 282–301.

[11] Some references to recent work on thermal stresses are given at the end of this paper. See [31–42].

4. Deflection of non-isotropic plates. The industrial demands for lighter and stronger materials have given a new impetus to a study of the behavior of various structural members made of non-isotropic materials. In particular, the need of an accurate knowledge of the performance of plates and shells made of plywood is strongly felt, and in this connection some extensions of the method outlined in §2 offer new possibilities. I shall indicate only one such extension which yields fruitful results in a study of small deflections of a thin non-isotropic plate subjected to a load $q(x, y)$ distributed normally over one of the faces of the plate. A very general type of anisotropy is assumed, and it will be supposed that the xy-plane of the coordinate system is the only plane of elastic symmetry.

It can be shown,[12] by utilizing the usual assumptions of the thin plate theory, that the differential equation governing small deflection w has the form:

$$b_{11}\frac{\partial^4 w}{\partial x^4} + 3b_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(b_{12} + b_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 3b_{26}\frac{\partial^4 w}{\partial x \partial y^3} + b_{22}\frac{\partial^4 w}{\partial y^4} = \frac{3q(x, y)}{2h^3}, \tag{4.1}$$

where the b_{ij} are the elastic constants and $2h$ is the thickness of the plate.

The equation (4.1) can be written in a symbolic form, with the aid of four linear operators, as

$$D_4 D_3 D_2 D_1 w = \frac{3q}{2h^3},$$

where $D_i = \partial/\partial y - \mu_i \partial/\partial x$, and the constants μ_i are the roots of the characteristic equation

$$b_{22}\mu^4 + 3b_{26}\mu^3 + 2(b_{12} + b_{66})\mu^2 + 3b_{16}\mu + b_{11} = 0. \tag{4.2}$$

It follows from the fact that the potential energy of any physically realizable state of stress is positive that the roots μ_i of the characteristic equation (4.2) are complex, and since the coefficients b_{ij} are real, they must have the forms:[13]

$$\mu_1 = \alpha + \beta i, \qquad \mu_2 = \gamma + \delta i, \qquad \mu_3 = \bar{\mu}_1, \qquad \mu_4 = \bar{\mu}_2,$$

where α, β, γ, δ are real and $\beta \neq 0$, $\delta \neq 0$.

[12] See [43, 44].

[13] A proof of this was given by S. N. Lechnitzky [43].

Accordingly, the general solution of equation (4.1), for the case of distinct roots,[14] can be written as

$$w = \sum_{k=1}^{4} F_k(x + \mu_k y) + w_0(x, y), \tag{4.3}$$

where $w_0(x, y)$ is any particular solution of (4.1), and the F_k are arbitrary functions. Since the deflection $w(x, y)$ is real, it follows that the functions $F_k(x+\mu_k y)$ must be conjugate in pairs, so that one can write the solution (4.3) in the form:

$$w = \phi(z_1) + \psi(z_2) + \overline{\phi(z_1)} + \overline{\psi(z_2)} + w_0(x, y), \tag{4.4}$$

where $\phi(z_1)$ and $\psi(z_2)$ are arbitrary functions of the complex variables

$$z_1 = x + \mu_1 y, \qquad z_2 = x + \mu_2 y. \tag{4.5}$$

Let the region corresponding to the middle surface of the unstrained plate be denoted by R and introduce two new systems of cartesian coordinates (x_1, y_1) and (x_2, y_2) related to the coordinates (x, y) by the formulas

$$z_1 = x_1 + iy_1, \qquad z_2 = x_2 + iy_2. \tag{4.6}$$

It follows from inspection of (4.5) and (4.6) that the three systems of coordinates (x, y), (x_1, y_1), and (x_2, y_2) are connected with one another by the equations

$$x_1 = x + \alpha y, \qquad y_1 = \beta y,$$
$$x_2 = x + \gamma y, \qquad y_2 = \delta y.$$

Hence to any point P in the region R there correspond points P_1 and P_2 in some regions R_1 and R_2 of the z_1- and z_2-planes, respectively, and it is obvious that the regions R_1 and R_2 are obtainable from the region R by a homogeneous deformation of the xy-plane. Thus, when the complex variable $z=x+iy$ varies in the region R, the variables z_1 and z_2 vary in the corresponding regions R_1 and R_2. Consequently, the functions $\phi(z_1)$ and $\psi(z_2)$ entering in the general solution (4.4) can be regarded as functions of the ordinary complex variables z_1 and z_2 defined in two different planes.

Ordinarily one is interested in determining the deflection w when

[14] In case of equal roots ($\mu_1=\mu_2$, $\mu_3=\mu_4$) the general solution assumes the form $w=(x+\mu_2 y)\ F_1(x+\mu_1 y)+(x+\mu_1 y)\ F_2(x+\mu_2 y)+F_3(x+\mu_1 y)+F_4(x+\mu_2 y)+w_0(x,\ y)$ and a similar discussion can be carried out for this special case. If the plate is isotropic the equation (4.1) assumes the form $\nabla^2\nabla^2 w=q/D$, where D is the flexural rigidity, and the general solution can be written as $w=\phi(z)+\bar{z}\psi(z)+\overline{\phi(z)}+z\overline{\psi(z)}+w_0(x, y)$.

the edges of the plate are subjected to a prescribed distribution of moments and shearing forces. Inasmuch as the moments and shearing forces are easily expressible in terms of the derivatives of the deflection function $w(x, y)$, one is led to a set of functional equations (appropriate to various modes of fixing plates at the edges) for the determination of the functions $\phi(z_1)$ and $\psi(z_2)$. In general, the functional equations are of the type

$$(4.7) \qquad a_{1j}\phi'(z_1) + a_{2j}\psi'(z_2) + \overline{a_{1j}\phi'(z_1)} + \overline{a_{2j}\psi'(z_2)} = A_j(s), \quad j = 1, 2,$$

where the a_{ij} are known constants and $A_j(s)$ are known functions prescribed on the contour C bounding the region R.

The problem of determining the functions $\phi(z_1)$ and $\psi(z_2)$ in this case is naturally more involved than in the corresponding isotropic problem, but the complications are not of a basic sort. One practical mode of solving a system of equations (4.7) is to introduce a pair of suitable mapping functions $z_1 = \omega_1(\zeta)$ and $z_2 = \omega_2(\zeta)$ transforming the regions R_1 and R_2 into the unit circle $|\zeta| \leqq 1$ in the auxiliary ζ-plane, and thereafter proceed in the same manner as indicated in §2.

Some interesting results pertaining to the problem of deflection of a clamped non-isotropic plate, subjected to a quite general distribution of normal load, were obtained recently by V. Morkovin in a partially completed doctoral dissertation. By utilizing the method outlined above, Morkovin obtained in closed form the solution of the problem of deflection of a clamped elliptical plate when the loading function $q(x, y)$ is a polynomial in x and y. The important case of a clamped circular plate follows directly from it by specializing the values of the parameters appearing in the solution.

5. **Concluding remarks.** The foregoing account of a somewhat novel use of the theory of functions of a complex variable was concerned mainly with the applications to two-dimensional problems of elasticity. However, the central idea of replacing the differential equation and boundary conditions by an equivalent functional equation is obviously capable of extension to problems in other branches of applied mathematics.

A consideration of the problems of equilibrium of cylindrical bodies, and especially those relating to a study of initially curved and twisted beams, shows that occasionally there are advantages in utilizing similar notions in some classes of three-dimensional problems [48–56].

The importance of the theory of functions of a complex variable in connection with the solution of the boundary value problems in-

volving Laplace's equation is, of course, recognized by every applied mathematician. However, even in the consideration of such problems as those of Neumann and Dirichlet, the engineers appear to be so absorbed in special devices and artifices that a simple formulation of such problems is often overlooked.

In order to set in relief the basic notions that underlie the foregoing formulation of the two-dimensional problems of elasticity, it may not be out of place to recall a simple approach to the problem of Dirichlet.

Let the function $U(x, y)$, continuous in the closed simply-connected region R and harmonic in the interior of R, assume continuous values $f(s)$ on the boundary C of R. Then the boundary condition can be written in the form

$$\phi(z) + \overline{\phi(z)} = 2f(s) \qquad \text{on } C,$$

where $\mathrm{Re}\phi(z) = U(x, y)$. Now, if the region R is mapped conformally on the unit circle $|\zeta| \leqq 1$ with the aid of the function $z = \omega(\zeta)$, then the foregoing boundary condition assumes the form

$$\phi_1(\sigma) + \overline{\phi_1(\sigma)} = 2f_1(\theta),$$

where $\sigma = e^{i\theta}$, and $\phi_1(\zeta)$ and $f_1(\theta)$ are the transforms of $\phi(z)$ and $f(s)$. Multiplying by $(1/2\pi i)d\sigma/(\sigma - \zeta)$ and integrating over the contour γ, yields

$$\frac{1}{2\pi i}\int_\gamma \frac{\phi_1(\sigma)}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\phi_1(\sigma)}}{\sigma - \zeta}\,d\sigma = \frac{1}{\pi i}\int_\gamma \frac{f_1(\theta)}{\sigma - \zeta}\,d\sigma,$$

and the evaluation of integrals gives at once

$$\phi_1(\zeta) = \frac{1}{\pi i}\int_\gamma \frac{f_1(\theta)}{\sigma - \zeta}\,d\sigma + \text{const.},$$

which is merely the formula of Schwarz.

The remarkable simplicity of the formulation of the two-dimensional problems of elasticity with the aid of functional equations is likely to produce an impression that the task of obtaining practically useful solutions is a straightforward matter. While it is true that in some cases the solution of rather difficult problems is obtained with remarkable ease and rapidity, in many instances severe calculational difficulties are present. These difficulties are connected mainly with the determination of the mapping function. However, recent advances in the development of practical methods of constructing map-

ping functions for simply- and multiply-connected domains justify some degree of optimism.

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UNIVERSITY OF WISCONSIN

ON EQUIVALENCE OF CERTAIN TYPES OF SERIES OF ORTHONORMAL FUNCTIONS

G. H. PEEBLES

If one has a set of "properly" independent functions $u_1(x), u_2(x), \cdots, u_n(x), \cdots$ and a function $p(x)$ which on an interval (a, b) is non-negative, integrable and such that $\int_a^b p(x)dx > 0$, one can construct a second set $v_1(x), v_2(x), \cdots, v_n(x), \cdots$ $[v_n(x) = a_{n1}u_1(x) + \cdots + a_{nn}u_n(x)$, where a_{nk} is a constant] whose members satisfy the relations

$$\int_a^b p(x)v_n(x)v_m(x)dx = \delta_{nm}.$$

Associated with such sets of orthonormal functions is the problem of expanding an "arbitrary" function $f(x)$ in a convergent series of the members of a particular set, that is, the problem of determining conditions sufficient for

$$\lim_{n\to\infty} \sum_{k=1}^{n} v_k(x) \int_a^b p(t)f(t)v_k(t)dt = f(x). \tag{1}$$

One approach to the problem lies in showing that expansion (1) is equivalent to another expansion, usually a Fourier series, whose behavior is known.[1] In the case of orthogonal polynomials, where $u_n(x) = x^{n-1}$, the writer was able to show, under conditions not too restrictive, that the expansions of a function in terms of the two sets of orthogonal polynomials corresponding, respectively, to different weight functions converge to the same value or diverge together.[2] It is the purpose of this note to point out that similar results can be obtained for systems of orthonormal functions constructed from a set $[u_n(x)]$ which has the property that the product of any two members

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[1] Theorems and references on equiconvergence for orthogonal polynomials are given by G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, chaps. 9 and 13. For equivalence for another type of orthogonal functions, see J. Mercer, *Sturm-Liouville series of normal functions in the theory of integral equations*, Philosophical Transactions of the Royal Society of London, (A), vol. 211 (1912), pp. 111–198; pp. 174–177.

[2] *An equivalence theorem for series of orthogonal polynomials*, Proceedings of the National Academy of Sciences, vol. 3 (1939), pp. 97–104.

$u_n(x)u_m(x)$ is a linear combination of the first $n+m+\beta$ members, where β may depend on n and m, but never exceeds some integer, say, j. A set with this property, for instance, underlies a set of polynomials in two variables orthogonal on an arc of an algebraic curve.[3] If the curve is the hyperbola $y^2=x^2-1$, the set $[u_n(x)]$ is $1, x, (x^2-1)^{1/2}, x^2, x(x^2-1)^{1/2}, \cdots$. The system of trigonometric functions $1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots$, which is essentially a set of polynomials orthogonal on a circle, will be used later for the purposes of illustration.

The proof of the equivalence of expansions is usually made to depend on asymptotic formulas for the orthogonal functions.[4] The method used here, however, rests chiefly on the familiar closure formula of Parseval and a fundamental property of orthonormal functions, namely, that $\int_a^b p(x)[\phi(x)-\sum_1^n c_k v_k(x)]^2 dx$ is a minimum with respect to the choice of the values of the c's when $c_k=\int_a^b p(x)\phi(x)v_k(x)dx$. In addition to the assumption that $u_n(x)u_m(x)$ is a linear combination of not more than the first $n+m+j$ members, it is convenient to suppose at the outset that the u's and all functions to be introduced are such that Parseval's closure formula holds and the necessary integrals exist whenever the argument so requires.

Let $[v_n(x)]$ and $[w_n(x)]$ be the sets of orthonormal functions constructed from the u's on the interval (a, b) and associated, respectively, with the weight functions $p(x)$ and $q(x)=r(x)p(x)$. Let

$$s_n(x) = \sum_{i=1}^{n} b_i v_i(x) = \sum_{i=1}^{n} v_i(x) \int_a^b p(t)f(t)v_i(t)dt$$

and

$$S_n(x) = \sum_{i=1}^{n} w_i(x) \int_a^b q(t)f(t)w_i(t)dt$$

be the corresponding nth order partial sums of the expansions of a function $f(x)$. The difference between the two sums is a linear combination of the u's, and so may be expressed in terms of the w's:

$$S_n(x) - s_n(x) = \sum_{k=1}^{n} r_{kn} w_k(x). \tag{2}$$

The coefficient

[3] See Dunham Jackson, *Orthogonal polynomials on a plane curve*, Duke Mathematical Journal, vol. 3 (1937), pp. 228–236; pp. 232–235; *Orthogonal polynomials in three variables*, ibid., vol. 4 (1938), pp. 441–454.

[4] See Footnote 1.

$$r_{kn} = \int_a^b q(x)w_k(x)[S_n(x) - s_n(x)]dx$$
$$= \int_a^b p(x)[r(x)w_k(x)][f(x) - s_n(x)]dx.$$

Hence, if one sets $B_{ik} = \int_a^b p(x)r(x)w_k(x)v_i(x)dx$, the assumption of the validity of Parseval's formula allows one to write

$$r_{kn} = \sum_{i=n+1}^{\infty} b_i B_{ik}$$

which with Cauchy's inequality yields the relation

$$|r_{kn}| \leqq \left\{\sum_{i=n+1}^{\infty} b_i^2\right\}^{1/2} \left\{\sum_{i=n+1}^{\infty} B_{ik}^2\right\}^{1/2}.$$

Since $\lim_{n\to\infty} \sum_{n+1}^{\infty} b_i^2 = 0$, it is clear from the last inequality and relation (2) that, if one writes $l_{nk} = \{\sum_{n+1}^{\infty} B_{ik}^2\}^{1/2}$, the existence of $L = \lim_{n\to\infty} \sum_{k=1}^{n} l_{nk}$ is sufficient for

$$\lim_{n\to\infty} [S_n(x) - s_n(x)] = 0$$

for any value of x for which $|w_k(x)|$ is bounded for all values of k.

The preceding result depends on Parseval's formula and not on the rule of combination assumed for the u's. A more serviceable form stating that L surely exists, if the ratio function $r(x)$ belongs to a suitable class of functions, is obtained when the special rule of combination holds.

The integral formula

$$(3) \qquad l_{nk}^2 = \sum_{i=n+1}^{\infty} B_{ik}^2 = \int_a^b p(x)\left[r(x)w_k(x) - \sum_{i=1}^{n} B_{ik}v_i(x)\right]^2 dx$$

provides a means for getting at conditions sufficient for the existence of L. An immediate consequence of (3) is that l_{nk} is bounded for all values of n and k, if $r(x)$ is bounded on (a, b); for, if C is an upper bound of $r(x)$, by Bessel's inequality for orthogonal functions,

$$l_{nk}^2 \leqq \int_a^b p(x)[r(x)w_k(x)]^2 dx = \int_a^b C\,r(x)p(x)[w_k(x)]^2 dx = C.$$

This result will be needed immediately.

Suppose $C \geqq r(x)$ and $r(x)$ is a linear combination of the u's of order m. Then $r(x)w_k(x)$ is a linear combination of the u's and consequently of the v's, of order not more than $m+k+j$. The integral of formula (3)

vanishes for $n \geqq m+k+j$; therefore $l_{nk}=0$ for $k \leqq n-(m+j)$, and L exists since always $\sum_{k=1}^{n} l_{nk} \leqq (m+j)C$.

Suppose that on (a, b) $C \geqq r(x) \geqq c>0$ and that $r(x)=\lim_{n\to\infty} G_n(x)$, where $G_n(x)$ is a linear combination of the u's with ϵ_n, the maximum or least upper bound of $|r(x)-G_n(x)|$ on (a, b), such that $\sum_1^\infty \epsilon_n$ is convergent. Since $w_k(x)G_{n-k-j}(x)$ is a linear combination of the v's of order n at most, one has for $k \leqq n-j-1$

$$l_{nk} \leqq \left\{ \int_a^b p(x)\left[w_k(x)r(x) - w_k(x)G_{n-k-j}(x)\right]^2 dx \right\}^{1/2}$$

$$\leqq \epsilon_{n-k-j} \left\{ \int_a^b p(x)\left[w_k(x)\right]^2 dx \right\}^{1/2}$$

$$\leqq \epsilon_{n-k-j} \left\{ C^{-1} \int_a^b p(x)r(x)\left[w_k(x)\right]^2 dx \right\}^{1/2} = C^{-1/2}\epsilon_{n-k-j}.$$

For the remaining $j+1$ values of k $(n-j, n-j+1, \cdots, n)$, $l_{nk} \leqq C^{1/2}$. Hence L exists.

The next assumption concerning the nature of $r(x)$ is very specific, but the result, when combined with that obtained in the case where $r(x)$ is a linear combination of the u's, may increase the generality of the results as an example will show later. On (a, b) suppose, unlike the last case, the lower bound of $r(x)$ is zero, but there exists an mth order linear combination of the u's $h_m(x)$ such that $C \geqq r(x) \geqq [h_m(x)]^2$ and such that $r(x)/h_m(x)$, like $r(x)$ in the preceding case, can be approximated by a set of G's. Then for $k \leqq n-m-2j-1$

$$l_{nk} = \left\{ \int_a^b p(x)\left[h_m(x)w_k(x)\right]^2\left[r(x)/h_m(x) - G_{n-m-k-2j}(x)\right]^2 dx \right\}^{1/2}$$

$$\leqq \epsilon_{n-m-k-2j} \left\{ \int_a^b p(x)r(x)\left[w_k(x)\right]^2 dx \right\}^{1/2} = \epsilon_{n-m-k-2j}.$$

As before, if $\sum_1^\infty \epsilon_n$ is convergent, L exists.

A review of the conditions imposed on $r(x)$ discloses that in no case is $r(x)$ allowed to become infinite and that only under very restrictive hypotheses is $r(x)$ allowed to vanish. If $r(x)$ were discontinuous, one would expect, in general, that approximating sums of the sort required would not exist. An example in trigonometric functions on the interval $(-\pi, \pi)$ will indicate briefly how and when the results just obtained may be used to circumvent difficulties presented by such exceptional points.

In the case of trigonometric functions, theorems due to Dunham Jackson provide the desired information about the approximating

sums.[5] For present purposes it suffices to know that a function of period 2π which possesses everywhere a kth order derivative satisfying a Lipschitz condition of order α can be approximated by a trigonometric sum of order n so closely that the absolute value of the error nowhere exceeds $K/n^{k+\alpha}$, K a constant. Hence, if $r(x) \geqq c > 0$ and has a derivative which satisfies a Lipschitz condition of order α and if $r(x)$ and $f(x)$ are of classes $L_p(-\pi, \pi)$ and $L_p^2(-\pi, \pi)$ so that Parseval's formula holds, one has equivalence between the expansions of $f(x)$ in the two sets of trigonometric sums corresponding to $p(x)$ and $r(x)p(x)$ wherever the sums associated with $r(x)p(x)$ are bounded. Suppose, however, that $r(x)$, a function of period 2π, is bounded, stays away from zero and has a second derivative everywhere in $(-\pi, \pi)$ except at $x=0$. At the origin, suppose $r(x)$ tends to vanish on the left and to become infinite on the right, but in such a way that $r(x) \geqq \sin^2 x$ everywhere and $\lim_{x=0+} \sin x\, p(x)$ exists. The function described has no particular merit except that it serves to illustrate how exceptional points may be handled. Let it again be assumed that $f(x)$ and $r(x)$ are of classes $L_p(-\pi, \pi)$ and $L_p^2(-\pi, \pi)$. It follows from the first case considered, where $r(x)$ is a linear combination of the u's, that L corresponding to the weight functions $\sin^{10} x\, r(x)p(x)$ and $r(x)p(x)$ exists. On the other hand $\sin^{10} x\, r(x) \geqq (\sin^6 x)^2$ and $\sin^{10} x\, r(x)/\sin^6 x$, on removing the discontinuity at the origin, has a second derivative everywhere and so can be approximated by an nth order trigonometric sum with the absolute value of the error not exceeding K/n^2. Hence, by virtue of the third case, L corresponding to the weight functions $\sin^{10} x\, r(x)p(x)$ and $p(x)$ also exists. Therefore the expansions of $f(x)$ corresponding to $r(x)p(x)$ and $p(x)$ are each equivalent to the expansion corresponding to $\sin^{10} x\, r(x)p(x)$, wherever the sums orthonormal with respect to the last weight function are bounded. It follows that the expansions associated with $r(x)p(x)$ and $p(x)$ are equivalent, wherever the sums orthonormal with respect to $\sin^{10} x\, r(x)p(x)$ are bounded.

The results on equivalence given here depend, in part, on the boundedness of the functions orthonormal with respect to $r(x)p(x)$. There is at hand in the foregoing discussion a result which may be useful in this connection. Referring to the definition of B_{ik}, one sees that formally

$$r(x)w_k(x) = \sum_{i=k-j}^{\infty} B_{ik} v_i(x) \tag{4}$$

[5] Dunham Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. 11, 1930, chap. 1.

and

$$v_i(x) = \sum_{k=1}^{i} B_{ik} w_k(x). \tag{5}$$

Since $B_{ik} \leqq \{\sum_{j=1}^{\infty} B_{jk}^2\}^{1/2} = l_{i-1,k}$, in all the cases of the ratio function $r(x)$ considered, the right-hand members of (4) and (5) are absolutely convergent and bounded, wherever, respectively, the v's and w's are bounded. Hence, if conditions are such that the right-hand member of (4) converges to the value of the left-hand member and if a set of points is known for which the v's are bounded, then the w's are bounded on the same set except where $r(x) = 0$. Similarly, boundedness of the w's leads through (5) to results on the boundedness of the v's.

UNIVERSITY OF MINNESOTA

A MAPPING CHARACTERIZATION OF PEANO SPACES

O. G. HARROLD, JR.

The Hahn-Mazurkiewicz theorem states that any Peano space (compact, connected, locally connected, metric space) is a continuous image of the interval $0 \leqq t \leqq 1$, and conversely. Clearly, the mapping function is not uniquely determined. If the Peano space $\mathfrak{M}$ has special topological properties, the mapping may be selected in a simpler fashion than might be expected generally. On the other hand, special properties of $\mathfrak{M}$ may impose certain necessary restrictions on the mapping. For example, if $\mathfrak{M}$ is a regular continuum in the sense of Menger, then, by a theorem due to Nöbeling,[1] there is a continuous mapping f of the circle[2] onto $\mathfrak{M}$ such that each point of finite order is covered by the mapping a number of times which does not exceed the order of the point. That is, if $o(x)$ is the order of the point x and $m(x)$ is the number of points in $f^{-1}(x)$, then $m(x) \leqq o(x)$ for each point for which $o(x)$ is finite. On the other hand, if $\mathfrak{M}$ is of dimension n, then *any* continuous mapping of a 1-dimensional compact set onto $\mathfrak{M}$,

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[1] G. Nöbeling, *Reguläre Kurven als Bilder der Kreislinie*, Fundamenta Mathematicae, vol. 20 (1933), pp. 30–46.

[2] The interval may be used instead of the circle if we make $f(0) = f(1)$ and count inverses on $0 \leqq t < 1$.

in particular, of an interval or circle, is such that there is a dense set of points in the n-dimensional part of $\mathcal{M}$ each of which has at least n inverse points in the original set.[3]

Denote the set of local separating points of the Peano space $\mathcal{M}$ by $\mathcal{L}$.[4] If $\mathcal{M} \subset \overline{\mathcal{M}-\mathcal{L}}$, that is, if $\mathcal{M}$ contains no free arcs,[5] there is a strongly irreducible mapping of the interval $\mathfrak{I}$ or circle $\mathcal{C}$ onto $\mathcal{M}$.[6] That is, for such spaces there exist continuous mappings of $\mathfrak{I}$ or $\mathcal{C}$ onto $\mathcal{M}$ such that no proper closed subset maps onto the whole space. Thus if $\mathcal{M}$ is an n-dimensional sphere and f is a strongly irreducible mapping of $\mathfrak{I}$ onto $\mathcal{M}$, there is a dense set of points each covered at least $n+1$ times and also a dense set of points each covered just once.

In addition to the symbols f, $\mathcal{M}$, $\mathcal{L}$, $m(x)$ and $o(x)$ used above, the following notations will be observed. Let ψ denote the aggregate of points $x \in \mathcal{M}$ lying in an open free arc of $\mathcal{M}-a-b$. If for a (continuous) mapping of $\mathfrak{I}$ into a subset of $\mathcal{M}$, $y \in \psi$ implies $m(y) \leqq 2$, the mapping will be said to be of type $\mathfrak{M}$.

THEOREM 1. *Let a and b be points of the Peano space $\mathcal{X}$. There is a continuous mapping of the interval $0 \leqq t \leqq 1$ onto $\mathcal{X}$ of type $\mathfrak{M}$ such that $f(0)=a$, $f(1)=b$.*

The theorem asserts, essentially, that there is a mapping of $\mathfrak{I}$ onto $\mathcal{X}$ such that every free arc is swept through at most twice.

The following lemmas will be useful in the proof of Theorem 1.

LEMMA 1.1. *If D is a subcontinuum of the dendrite D^0, to $e>0$ there is a finite collection $D^1, D^2, \cdots, D^n$ of dendrites in D^0 such that $D=D^1 \subset D^2 \subset \cdots \subset D^n=D^0$ and each component of $D^{i+1}-D^i$ has a diameter less than or equal to e.*

Let ρ be a convex metric[7] on D^0. Let $d=$ glb of numbers r such that

[3] If the original set is locally euclidean, the phrase *at least n* may be replaced by *at least $n+1$*. See W. Hurewicz, *Über dimensionserhohende stetige Abbildungen*, Journal für die reine und angewandte Mathematik, vol. 169 (1933), pp. 71–78.

[4] The point p is called a local separating point of M provided that to every neighborhood U of p there is some pair of points of the component of U containing p which is separated in $U-p$.

[5] The set A is called a free arc of M provided A is an arc and the interior of A is open in M. An *open* free arc is an open subset of M which is homeomorphic to $0<x<1$. A point is said to lie in an open free arc provided there is a neighborhood of the point in M which is an open free arc. It is to be noted that if M is an arc, neither end point lies in an open free arc.

[6] O. G. Harrold, Jr., *A note on strongly irreducible maps of an interval*, Duke Mathematical Journal, vol. 6 (1940), pp. 750–752.

[7] The metric ρ is called *convex* after Menger provided that to each pair of distinct points x and y in M there is a point of $M-x-y$ such that $\rho(x, z)+\rho(z, y)=\rho(x, y)$.

$D^0 \subset S(D, r)$. Each of the sets $D^i = S[D, (i-1)e]$, $i = 1, 2, \cdots, n$ is a subcontinuum of D^0. The sets D^i satisfy our requirements, where n is the smallest integer such that $(n-1) \geqq d$.

LEMMA 1.2. *Let $\mathfrak{X}$ be a Peano space. There is a sequence (T_i) of dendritic graphs in $\mathfrak{X}$ such that* (a) $\lim T_i = \mathfrak{X}$, (b) $T_{i+1} \supset T_i$, *and* (c) *each component of $T_{i+1} - T_i$ has a diameter less than or equal to $1/2^{i+1}$.*

That a sequence of dendrites exists in $\mathfrak{X}$ satisfying (a) and (b) is well known. Application of Lemma 1.1 to the successive terms of this sequence gives the desired result.

Proof of Theorem 1. The theorem is true for a connected dendritic graph with n end points, as a simple induction shows. Suppose, temporarily, that neither a nor b lies in an open free arc. Let T_1 be a dendritic graph in $\mathfrak{X}$ containing a and b:[8] Let f_1 denote a continuous mapping of type $\mathfrak{M}$ of 3 onto T_1 with $f_1(0) = a$, $f_1(1) = b$. Let (T_i) be a sequence of dendritic graphs satisfying Lemma 1.2. Since T_2 is a graph, $T_2 - T_1$ has but a finite number of components which may be denoted by $C_1^1, C_2^1, \cdots, C_{n_1}^1$. Each $\overline{C_i^1} \cdot T_1$ is a point c_i. Let $x_i \in f_1^{-1}(c_i)$. By a rearrangement of notation it may be supposed that $0 \leqq x_1 < x_2 < \cdots < x_{p_1} \leqq 1$, $p_1 \leqq n_1$, where each x_i corresponds to a distinct c_i. Set $d_1 = \min |x_i - x_j|$, $i \neq j$, $|x_i|$, $x_i \neq 0$, $|1 - x_i|$, $x_i \neq 1$. To $\epsilon = 1/2$ there is a $d_2 > 0$ such that $|x - y| < d_2$ implies $\rho[f_1(x), f_1(y)] < \epsilon/2 = 1/2^2$, where ρ denotes the metric of $\mathfrak{X}$. Put $W = S(x_1 + x_2 + \cdots + x_{p_1}, d/3)$, where $d = \min(d_1, d_2)$. Let J_i be the component of W containing x_i. Let $I_1, I_2, \cdots, I_{p_1+1}$ be the intervals on 3 complementary to W, where I_1 becomes degenerate if $x_1 = 0$ and I_{p_1+1} degenerate if $x_{p_1} = 1$. The interval 3 is now subdivided into the intervals (in order) $I_1, \overline{J}_1, I_2, \overline{J}_2, \cdots, I_{p_1+1}$. Let t denote the piecewise linear map obtained by sending I_1 onto $(0x_1)$ with order preserved, I_2 onto $(x_1x_2), \cdots, I_{p_1+1}$ onto $(x_{p_1}1)$. For $x \in \sum I_i$, put $f_2(x) = f_1[t(x)]$. On $\overline{J}_i$ define a map g_i of the desired type so that $g_i(\overline{J}_i) = D_i$. where D_i is the enclosure of all components C_i^1 having c_i as a limit point. The set D_i is a dendrite of diameter less than or equal to $1/2^2$. The map g_i may be so selected that for the end points of J_i, $g_i = f_2$.

See K. Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), pp. 81 ff. For the existence of the metric assumed here, see C. Kuratowski and Whyburn, *Sur les elements cycliques et leurs applications*, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331.

[8] T_1 could be taken to be an arc but in order that the discussion to follow be general it is assumed only to be a connected linear graph containing no simple closed curve.

The definition of f_2 is now completed: if $x \in \mathfrak{I}_i$, $f_2(x) = g_i(x)$. Clearly, $f_2(\mathfrak{I}) = T_2$ and f_2 is continuous. For $x \in \sum I_i$, $|x - t(x)| < d$, hence $\rho[f_1(x), f_2(x)] < 1/2^2$. If $x \in \mathfrak{I}_i$, $\rho[f_1(x), f_1(x_i)] < 1/2^2$, and since the diameter of D_i is less than or equal to $1/2^2$, $\rho[f_2(x), f_2(t^{-1}(x_i))] < 1/2^2$. But $f_2(t^{-1}(x_i)) = f_1(x_i)$, hence by the triangle inequality $\rho[f_1(x), f_2(x)] < 1/2$. Thus if σ denotes the usual metric of the function space $\mathfrak{X}^{\mathfrak{I}}$, $\sigma(f_1, f_2) \leqq 1/2$.[9]

To show that f_2 is of type $\mathfrak{M}$ consider a point $y \in \psi \cdot T_2$. It is to be shown that $f_2^{-1}(y)$ contains at most two points, that is, $m(y, f_2) \leqq 2$. If $y \in T_2 - T_1$, y lies in an unique D_i, hence $m(y, f_2) \leqq 2$. If $y \in T_1 - \sum D_i$, $f_2^{-1}(y) = t^{-1}f_1^{-1}(y)$. Since f_1 has the desired property and t is 1-1 on the set composed of the interiors of the intervals I_i, $m(y, f_2) \leqq 2$. Consider the remaining case $y = c_i$. Here $m(y, f_1) = 1$, for suppose, on the contrary, $f_1^{-1}(y) \supset q_1 + q_2$. Since $y \in \psi$, $y \neq a$, b, hence q_1 and q_2 divide $\mathfrak{I}$ into three subintervals A, B and C. But each of these subintervals has as an image under f_1 a nondegenerate continuum containing y. Hence points of T_1 near y have three inverses on $\mathfrak{I}$, which denies the property of f_1. Since $y \in \psi$, only one component $C_i' = D_i$ of $T_2 - T_1$ can have y as a limit point. The mapping g_i has the desired property, thus $f_2^{-1}(y) = g_i^{-1}(y) + f_1^{-1}[t^{-1}(y)]$ is precisely a pair of points. Hence for $y \in \psi \cdot T_2$, $m(y, f_2) \leqq 2$, and f_2 is of type $\mathfrak{M}$.

The general inductive hypothesis is now clear.

To the dendrite T_n there is a continuous mapping f_n, $f_n(\mathfrak{I}) = T_n$, of type $\mathfrak{M}$ and such that $f_n(0) = a$, $f_n(1) = b$. Further, $\sigma(f_{i-1}, f_i) \leqq 1/2^{i-1}$, $i = 2, 3, \cdots, n$. The construction of f_{n+1} from f_n is accomplished precisely as above.

There is thus determined a sequence of points (f_n) of the space $\mathfrak{X}^{\mathfrak{I}}$ such that to $e > 0$ there is an index N such that for $i, j > N$, $\sigma(f_i, f_j) < e$. The space $\mathfrak{X}^{\mathfrak{I}}$ being complete, let $\lim f_n = f$. Clearly, $f(\mathfrak{I}) = \mathfrak{X}$. To complete the proof of Theorem 1 it will suffice to show that $y \in \psi$ implies $f^{-1}(y)$ has at most two components. For, if we grant this, the factorization $f = k[h(x)]$, where h is the monotone transformation obtained by shrinking the components of $f^{-1}(y)$ into points and k is the corresponding light transformation, yields $k(\mathfrak{I}) = \mathfrak{X}$, $m(y \in \psi, k) \leqq 2$, hence k is of type $\mathfrak{M}$.[10]

Suppose, on the contrary, $y \in \psi$ and X_1, X_2 and X_3 are three components of $f^{-1}(y)$. Since $y \in \psi$, $y \neq a$, b, hence $\mathfrak{I} - \sum X_i$ has precisely 4 components R_i, $i = 1, 2, 3$ and 4. Suppose w_i and t_i are the left and right end points of X_i, respectively. (If X_i is a point, $w_i = t_i$.) Let the

[9] If $f, g \in \mathfrak{X}^{\mathfrak{I}}$, $\sigma(f, g) = \text{lub } \rho[f(x), g(x)]$, $x \in \mathfrak{I}$.

[10] This is an application of a factor theorem for continuous transformations due to Eilenberg and Whyburn.

notation be arranged so that $w_i = \overline{R}_i \cdot X_i$, $i = 1, 2$ and 3. The points w_i and w_j, $i \neq j$, are, of course, distinct. Let A_i, $i = 1, 2$ and 3 be a subinterval of $\overline{R}_i$ containing w_i such that $f(A_i) \subset U$, where U is any fixed open free arc of $\mathfrak{X}$ containing y. Let B_{i+1} $i = 1, 2$ and 3 be a proper subinterval of $R_{i+1} - A_{i+1}$ containing t_i such that $f(B_i) \subset U$, where $A_4 = 0$ by definition. Since X_i is a component of $f^{-1}(y)$, the sets of $f(A_i)$, $f(B_i)$ are nondegenerate subarcs of U with at least the point y common. Some point y^0 of $U - y$ must be covered by at least three of these six sets. Denote three of the corresponding sets A_i (B_i), $i = 1, 2$ and 3 by C_1, C_2 and C_3. One end point of C_i, say a_i, maps into y. As C_i is traversed from a_i let b_i be the first point in $f^{-1}(y^0)$. Let G denote any subarc of $U - y - y^0$ which lies between y and y^0. Set $d = \rho(G, y + y^0) > 0$. Then for n large enough $\sigma(f_n, f) < d/3$. But $f_n(a_i b_i)$ is a connected subset of U which contains a point from each component of $U - G$, hence $f_n(a_i b_i) \supset G$. This denies that f_n is of type $\mathfrak{M}$. The proof of Theorem 1 under the special restriction that neither a nor b lies in a free arc has been completed.

To remove the restriction suppose first that only a lies in a free arc of $\mathfrak{X}$. Imagine that $\mathfrak{X}$ is situated in the Hilbert cube and let $\mathcal{A}$ be an arc which is joined onto $\mathfrak{X}$ at a and has no other point in $\mathfrak{X}$. Let a^1 be the other end point of $\mathcal{A}$. Construct a mapping as above with $f(0) = a^1$, $f(1) = b$, $f(\mathfrak{I}) = \mathfrak{X} + \mathcal{A}$. Since neither a^1 nor b lies in an open free arc of $\mathfrak{X} + \mathcal{A}$, such a mapping will exist. Let x^1 be the least x for which $f(x) = a$. Then the mapping f on the interval $x^1 \leqq t \leqq 1$ satisfies our requirements. A similar modification suffices to treat the case in which b is an open free arc and also the case in which both a and b have this property.

Set $\mathcal{W} = \mathfrak{X} - \bar{\psi}$. The set $\mathcal{W}$ is open. Put $\mathcal{Q} = \psi + (\mathcal{W} \cdot \mathcal{N})$, where $\mathcal{N}$ is the set of nonlocal separating points of $\mathfrak{X}$. We come now to the principal result.

THEOREM 2. *Let a and b be points of the metric space $\mathfrak{X}$. In order that $\mathfrak{X}$ be a Peano space it is necessary and sufficient that for any countable subset $\mathcal{P}$ of $\mathcal{Q} - a - b$ there be a continuous mapping f of $0 \leqq t \leqq 1$ onto $\mathfrak{X}$ such that $f(0) = a$, $f(1) = b$ and $y \in \mathcal{P}$ implies $m(y) \leqq 2$.*

PROOF. The sufficiency is clear. If $\psi = 0$, the result is known, in fact, in this case a mapping of the described type exists such that for $y \in \mathcal{P}$, $m(y) = 1$.[6] It is supposed, then, that $\psi \neq 0$. By application of Theorem 1, there is a mapping of type $\mathfrak{M}$ of $\mathfrak{I}$ onto $\mathfrak{X}$ with $f(0) = a$, $f(1) = b$. The desired map will be obtained by a modification of f.

To facilitate the discussion it will be supposed that $\mathfrak{X}$ has an S-metric, that is, a metric ρ such that for each $r > 0$ and $x \in \mathfrak{X}$, $\overline{S}(x, r)$ is a lo-

cally connected continuum.[11] Let $A=a_1+a_2+\cdots=\mathcal{P}\cdot\mathcal{W}=\mathcal{P}\cdot\mathcal{W}\cdot\mathcal{N}$. Set $d_1=\rho(a_1,\mathcal{X}-\mathcal{W})$. Choose a number e_1 such that $d_1/2<e_1<d_1$ and $A\cdot\{\overline{S}(a_1,\ e_1)-S(a_1,\ e_1)\}=0$. This is possible since A is countable. Let a_{k_2} be the first point of A in $\mathcal{W}-\overline{S}_1$, where $S_1=S(a_1,\ e_1)$. Let $d_2=\rho[a_{k_2},(\mathcal{X}-\mathcal{W})+S_1]$. The number e_2 is chosen so that $d_2/2<e_2<d_2$ and $A\cdot\{\overline{S}(a_{k_2},\ e_2)-S(a_{k_2},\ e_2)\}=0$. Continuing in this way a sequence of spheres (S_i) is determined such that (a) $\overline{S}_i$ is a Peano space, (b) $\overline{S}_i\cdot\overline{S}_j=0$, $i\neq j$, (c) $\delta(S_i)\to 0$, (d) $\sum\overline{S}_i=\overline{\mathcal{W}}$ and (e) $A\cdot(\overline{S}_i-S_i)=0$.

Set $V_i=f^{-1}(S_i)$. Let V_{ij}, $j=1,\ 2,\ \cdots$ be the components of V_i. Let V_{i1} be a component of V_i such that $V_{i1}\cdot f^{-1}(a_{k_i})\neq 0$. Every point of $\overline{S}_i$ is either a nonlocal separating point of $\overline{S}_i$ or a limit point of such points. This is clear if $x\in S_i$, for $S_i\subset\mathcal{W}$. If $x\in\overline{S}_i-S_i$, x is a limit point of points of S_i and hence a limit point of nonlocal separating points of $\overline{S}_i$. Thus, having shown that $\overline{S}_i$ is a Peano space with no free arcs, there is a strongly irreducible mapping, $f_{i1}(V_{i1})=\overline{S}_i$, such that $f=f_{i1}$ on $\overline{V}_{i1}-V_{i1}$ and $y\in\mathcal{P}\cdot\overline{S}_i$ implies $f_{i1}^{-1}(y)$ is a single point.

On V_{ij}, $j>1$, two cases are distinguished according as f maps the end points of V_{ij} into the same point or not. If f carries the end points of V_{ij} into x, define $f_{ij}\equiv x$ on $\overline{V}_{ij}$. If f carries the end points of V_{ij} into distinct points x and y, proceed as follows. The set $\overline{S}_i-A$ is a connected and locally connected G_δ set[12] in a complete space, hence there is an arc $R_{ij}\subset\overline{S}_i-A$ which joins x and y.[13] On $\overline{V}_{ij}$ define f_{ij} to be a homeomorphism into R_{ij} such that f_{ij} agrees with f on $\overline{V}_{ij}-V_{ij}$.

The new mapping g will now be defined. On $\mathfrak{I}-\sum V_{ij}$, set $g(x)\equiv f(x)$. On V_{ij}, set $g(x)=f_{ij}(x)$. Since f agrees with g on the end points of V_{ij} and each f_{ij} is continuous, g is continuous (we use here the condition (c) on the spheres (S_i)). Clearly, $g(\mathfrak{I})=\mathcal{X}$. If $y\in\mathcal{P}\cdot\psi$, $m(y,f)=m(y,g)\leqq 2$, by virtue of the fact that f is of type $\mathfrak{M}$. If $y\in\mathcal{P}\cdot\mathcal{W}$, y lies in an unique S_i and $g^{-1}(y)=f_{i1}^{-1}(y)$, hence $m(y,g)=1$.

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[11] J. L. Kelley, *A metric connected with property S*, American Journal of Mathematics, vol. 61 (1939), pp. 764–768.

[12] The complement of a countable set of nonlocal separating points in a Peano space is connected and locally connected, see G. T. Whyburn, *Semi-closed sets and collections*, Duke Mathematical Journal, vol. 2 (1936), pp. 685–690.

[13] This is the well known Moore-Menger generalization of the arcwise connectivity theorem for regions in a Peano space.

GENERALIZATIONS OF THE BERNOULLI POLYNOMIALS AND NUMBERS AND CORRESPONDING SUMMATION FORMULAS

TOMLINSON FORT

The Bernoulli polynomials and numbers have been generalized by Nörlund[1] to the Bernoulli polynomials and numbers of higher order. The Bernoulli numbers have been generalized by Vandiver. Analogous polynomials and sets of numbers have been defined from time to time, witness the Euler polynomials and numbers and the so-called Bernoulli polynomials of the second kind.[2]

In the present paper a generalization is made which includes all the above and many other interesting classes of polynomials and corresponding sets of numbers. As a matter of fact the definition of new classes of polynomials by the processes of this paper is a simple matter. In this connection particular attention is called to 2, (d), (h), (i), (j), (k), (l).

An important part of the paper is the development of a whole category of summation formulas related to the studied polynomials as the classical Euler-Maclaurin[3] formula is related to the Bernoulli polynomials or as Taylor's formula is related to $(x-a)^n$.

It will be observed that the work could be varied in detail resulting in closely related polynomials and numbers to those which are obtained. The particular procedure adopted is chosen so as to generalize the Bernoulli polynomials and numbers as now usually defined.[4]

1. **Definition of the polynomials and numbers.** Let us be given two linear operators P and Q with their inverses P^{-1} and Q^{-1}. We shall assume that P reduces the degree of any polynomial by 1 and that Q reduces the degree of any polynomial by $k \geqq 0$, that P operating on a constant gives zero and that Q operating on any polynomial of lesser degree than k gives zero. We assume that P, P^{-1}, Q, Q^{-1} each, where

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[1] N. E. Nörlund, *Differenzenrechnung*, p. 119. For other generalizations see H. S. Vandiver, Proceedings of the National Academy of Sciences, vol. 23, p. 555. See also Leonard Carlitz, this Bulletin, abstract 47-7-296, also abstract 47-9-358.

[2] See C. Jordan, *Calculus of Finite Differences*, p. 265.

[3] For usage of the name Euler-Maclaurin see footnote of a paper by the author this Bulletin, vol. 45 (1939), p. 748. Usage in the present instance is that of Nörlund, loc. cit., p. 29.

[4] See, for example, Nörlund, loc. cit., p. 17.

applicable, gives a unique result except that it is permitted that the result of operating with P^{-1} lack in uniqueness by an arbitrary additive constant and by Q^{-1} by an additive arbitrary polynomial of degree less than k.

We assume, moreover, that we are given a set of polynomials $f_n(x)$ where $f_n(x)$ is of degree n, such that

$$f_0(x) \equiv 1, \tag{1}$$

$$Pf_n(x) = nf_{n-1}(x), \tag{2}$$

$$\begin{aligned} &Q^{-1}n(n-1)\cdots(n-k+1)f_{n-k}(x) \\ &= {}_nL_0 f_n(x) + n\,{}_nL_1 f_{n-1}(x) + \frac{n(n-1)}{2}\,{}_nL_2 f_{n-2}(x) + \cdots \\ &\quad + C_{n,k}\,{}_nL_{n-k} f_k(x) + c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k, 0 \leqq k \leqq n, \end{aligned} \tag{3}$$

where ${}_nL_0, \cdots, {}_nL_{n-k}$ are independent of x but uniquely determined. The c's may be determined constants or may be arbitrary depending upon the nature of the operator Q^{-1}.

From (3)

$$\begin{aligned} &\frac{1}{n} PQ^{-1}n(n-1)\cdots(n-k+1)f_{n-k}(x) \\ &= {}_nL_0 f_{n-1}(x) + (n-1){}_nL_1 f_{n-2}(x) + \frac{(n-1)(n-2)}{2!}\,{}_nL_2 f_{n-3}(x) \\ &\quad + \cdots + C_{n-1,k}\,{}_nL_{n-k} f_{k-1}(x) \\ &\quad + [\text{terms of degree less than } k-1]. \end{aligned} \tag{4}$$

However,

$$\begin{aligned} &\frac{1}{n} Q^{-1}Pn(n-1)\cdots(n-k+1)f_{n-k}(x) \\ &= Q^{-1}(n-1)(n-2)\cdots(n-k)f_{n-k-1}(x) \\ &= {}_{n-1}L_0 f_{n-1}(x) + (n-1)\,{}_{n-1}L_1 f_{n-2}(x) \\ &\quad + \frac{(n-1)(n-2)}{2!}\,{}_{n-1}L_2 f_{n-3}(x) + \cdots + C_{n-1,k}\,{}_{n-1}L_{n-k} f_{k-1}(x) \\ &\quad + [\text{terms of degree less than } k]. \end{aligned} \tag{5}$$

We next assume

$$PQ^{-1}f_{n-k}(x) \equiv Q^{-1}Pf_{n-k}(x) + [\text{terms of degree less than } k].$$

Under this assumption

$$
\begin{aligned}
&{}_nL_0 f_{n-1}(x) + (n-1)\,{}_nL_1 f_{n-2}(x) + \frac{(n-1)(n-2)}{2!}\,{}_nL_2 f_{n-3}(x) \\
&\quad + \cdots + C_{n-1,k}\,{}_nL_{n-k} f_{k-1}(x) \\
(6)\qquad &\equiv {}_{n-1}L_0 f_{n-1}(x) + (n-1)\,{}_{n-1}L_1 f_{n-2}(x) \\
&\quad + \frac{(n-1)(n-2)}{2!}\,{}_{n-1}L_2 f_{n-3}(x) + \cdots + C_{n-1,k}\,{}_{n-1}L_{n-k} f_{k-1}(x) \\
&\quad + [\text{terms of degree less than } k].
\end{aligned}
$$

Equating coefficients we find

$$
{}_nL_0 = {}_{n-1}L_0,\ {}_nL_1 = {}_{n-1}L_1,\ {}_nL_2 = {}_{n-1}L_2, \cdots .
$$

In other words, the L's are independent of n. Henceforth we write simply $L_0, L_1, L_2, \cdots$. We let

$$
\begin{aligned}
&F_n(x) = L_0 f_n(x) + nL_1 f_{n-1}(x) \\
(7)\qquad &\quad + \frac{n(n-1)}{2!}\,L_2 f_{n-2}(x) + \cdots + L_n f_0(x), \qquad n \geqq 0.
\end{aligned}
$$

By (3)

$$
(8)\qquad QF_n(x) = n(n-1)\cdots(n-k+1) f_{n-k}(x), \qquad n \geqq k.
$$

For reasons of symmetry and consequent simplification of the sequel we do not take the L_n's just defined as our fundamental sequence of numbers but the set g_n determined from the following equations $f_n(g) = L_n$ where subscripts are applied to g rather than exponents in the expansion of $f_n(g)$. Such a determination is always possible and unique. We then write

$$
\begin{aligned}
&F_n(x) = f_0(g) f_n(x) + n f_1(g) f_{n-1}(x) \\
(9)\qquad &\quad + \frac{n(n-1)}{2!}\,f_2(g) f_{n-2}(x) + \cdots + f_n(g) f_0(x), \qquad n \geqq 0.
\end{aligned}
$$

We choose to write this $f_n(x+g)$ of which more will be said later. We then have

$$
(10)\qquad F_n(x) = f_n(x+g),
$$

$$
(11)\qquad PF_n(x) = nF_{n-1}(x), \qquad n \geqq 1.
$$

The polynomials $F_n(x)$ are the polynomials[5] in which we are interested

[5] If we replace (2) by $Pf_n(x) = n(n-1)\cdots(n-k+1) f_{n-k}(x)$ and add the requirement that the result of operating with P on any polynomial of degree less than k is zero, we are led to sequences of polynomials each of degree differing from that of the previous by k, a somewhat more general situation than that treated in the text.

and the numbers g_n constitute the corresponding sequence of numbers.

2. **Special cases.**[6] We now consider these special cases.

(a) Bernoulli polynomials and numbers:

$$P = \frac{d}{dx}, \qquad Q = \Delta, \qquad f_n(x) = x^n.$$

(b) Bernoulli polynomials and numbers of the second kind:

$$P = \underset{\omega}{\Delta}, \quad Q = \frac{d}{dx}, \quad f_n(x) = x^{(n)} = x(x-\omega)\cdots(x-(n-1)\omega).$$

(c) Bernoulli polynomials and numbers of higher order as defined by Nörlund:

$$P = \frac{d}{dx}, \qquad Q = \underset{\omega_1,\cdots,\omega_k}{\Delta^k}, \qquad f_n(x) = x^n, \qquad k > 1.$$

(d) Bernoulli polynomials and numbers of the second kind of higher order:[7]

$$P = \underset{\omega}{\Delta}, \qquad Q = \frac{d^k}{dx^k}, \qquad f_n(x) = x^{(n)}, \qquad k > 1.$$

(e) Bernoulli polynomials with Bernoulli numbers of higher order as defined by Vandiver:

$$P = \frac{d}{dx}, \qquad Q = \underset{\omega_1,\cdots,\omega_k}{\Delta^k}, \qquad f_n(x) = (x+l)^n.$$

Vandiver's numbers are the L_n's of the text rather than the g_n's.

(f) Euler polynomials:

$$P = \frac{d}{dx}, \qquad Q = M, \qquad f(x) = x^n.$$

---

[6] Certain notation in this section is as follows:

$$\Delta f(x) = f(x+1) - f(x); \qquad \underset{\omega}{\Delta} f(x) = \frac{1}{\omega}(f(x+\omega) - f(x));$$

$$\underset{\omega_1,\cdots,\omega_k}{\Delta} f(x) = \underset{\omega_1}{\Delta}\,\underset{\omega_2}{\Delta}\cdots\underset{\omega_k}{\Delta} f(x); \qquad Mf(x) = \tfrac{1}{2}(f(x+\omega) + f(x));$$

$$Tf(x) = \frac{f(qx) - f(x)}{qx - x}, \qquad q \neq 1.$$

[7] This name is applied through analogy with the Bernoulli polynomials of higher order as defined by Nörlund and as a natural extension of the Bernoulli polynomials of the second kind already referred to.

(g) Euler polynomials of higher order:

$$P = \frac{d}{dx}, \qquad Q = M^k, \qquad f(x) = x^n.$$

(h)[8] $$P = \frac{d}{dx}, \qquad Q = T^k, \qquad f(x) = x^{(n)}, \qquad k > 0.$$

(i) $$P = \frac{d}{dx}, \qquad Q = \underset{\omega_1,\cdots,\omega_k}{\Delta^k},$$

$f_n(x)$ any set of Appell[9] polynomials.

(j) $$P = \Delta, \qquad Q = \frac{d^k}{dx^k},$$

$f_n(x)$ any set of Appell polynomials of the second kind.

(k) $$P = a_0D^r + a_1D^{r-1} + \cdots + a_{r-1}D \equiv R(D),$$

where $D=d/dx$ and $a_0, a_1, \cdots, a_{r-1}$ are constants; $a_{r-1}\neq 0$, $Q=b_0\Delta^r + b_1\Delta^{r-1} + \cdots + b_s$, $b_0, b_1, \cdots, b_s$ being constants. $f_n(x)$ are polynomials obtained by successive solutions of the equation

$$R(D)f_n(x) = nf_{n-1}(x).$$

(l) Interchange D and Δ in (j).

3. **Summation formulas.** In the remainder of this paper we shall assume $f_n(0)=0$.

We first prove a lemma with reference to $f_n(x)$. We shall prove that

$$f_n(x+h) = f_n(x) + nf_{n-1}(x)f_1(h) + \frac{n(n-1)}{2!}f_{n-2}(x)f_2(h) + \cdots + f_n(h). \tag{12}$$

To do this we write

$$f_n(x+h) = b_0(x) + b_1(x)f_1(h) + b_2(x)f_2(h) + \cdots + b_n(x)f_n(h), \tag{13}$$

where the b's are as yet undetermined. Such an expression is possible

[8] F. H. Jackson, Quarterly Journal of Mathematics, vol. 41 (1909–1910), p. 195.

[9] We understand by a set of Appell polynomials a set of polynomials such that $(d/dx)f_n(x)=nf_{n-1}(x)$ and of the second kind such that

$$\underset{\omega}{\Delta} f_n(x) = nf_{n-1}(x).$$

since $f_n(x+h)$ is a polynomial of the nth degree in h. We determine the b's by successively applying the operator P to (13) as a function of h and then letting $h=0$ remembering that $f_n(0)=0$.

We have attached a meaning to $f_n(x+g)$. With this meaning in mind we prove that

$$f_n((x+h)+g) = f_n(x+(h+g)). \tag{14}$$

Exponents are applied to x and h, subscripts to g. Expansion of the left-hand member of (15) is the same as that given in (9) with $(x+h)$ replacing x. The whole expansion carried out on both sides of (12) is exactly the same as the expansions of $((x+h)+g)^n$ and $(x+(h+g))^n$ with the exponents replaced by the f-function. Thus $f_k(x)$ replaces x^k. Inasmuch as the binomial expansion yields an identity so does the expansion in terms of f that we are considering.

Now consider any polynomial of degree $m+k$ which we write

$$\begin{aligned}\psi(g) &= a_0 + a_1 f_1(x) + \cdots + a_{m+k} f_{m+k}(x),\\ \psi(x+g) &= a_0 + a_1 f_1(x+g) + \cdots + a_{m+k} f_{m+k}(x+g)\\ &\equiv a_0 + a_1 F_1(x) + \cdots + a_{m+k} F_{m+k}(x).\end{aligned}$$

Hence by (8) $Q\psi(x+g) = a_k(k!) + a_{k+1}(k+1)\cdots 2f_1(x) + \cdots + a_{m+k}(m+k)(m+k-1)\cdots(m+1)f_m(x)$. That is, $Q\psi(x+g) = P^k\psi(x)$, $m+k \geqq 0$. Similarly

$$Q\psi(x+h+g) = P^k\psi(x+h). \tag{15}$$

We apply Taylor's formula symbolically to this, that is, we write

$$Q\psi(x+h+g) = c_0(x) + c_1(x)F_1(h) + \cdots + c_m(x)F_m(h)$$

and determine the c's by first letting $h=-g$ and then successively applying P, remembering (10) and letting $h=-g$ after each application.[10] We get

$$Q\psi(x+h+g) = Q\psi(x) + \sum_{v=1}^{m} \frac{F_v(h)}{v!} P^v Q\psi(x).$$

Hence by (15)

$$P^k\psi(x+h) = Q\psi(x) + \sum_{v=1}^{m} \frac{F_v(h)}{v!} P^v Q\psi(x). \tag{16}$$

This is a polynomial identity. Let $\psi(x) = P^{-k}\chi(x)$ where some particular determination is chosen in case P^{-k} is not unique. Then

$$\chi(x+h) = \sum_{v=0}^{m} \frac{F_v(h)}{v!} P^v Q P^{-k}\chi(x). \tag{17}$$

[10] Here, of course, when h is replaced by g an exponent is changed to a subscript.

Like (16) this is a polynomial identity. It includes as special cases Taylor's formula, the Euler-Maclaurin formula[11] of classical mathematics, the Boole formula, and so on.

Formula (17) will be written out in detail for two interesting special cases.

Let

$$P = \frac{d}{dx}, \qquad Q = \underset{\omega}{\Delta}^k, \qquad P^{-1} = \int_0^x dx, \qquad k = 1, 2, \cdots;$$

$$\begin{aligned}\chi(x+h) = {} & \underset{\omega}{\Delta}^k \int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} \chi(x_k)\,dx_1 dx_2 \cdots dx_k \\ & + B_1^{(k)}(h)\underset{\omega}{\Delta}^k \int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} \chi(x_{k-1})\,dx_1 dx_2 \cdots dx_{k-1} \\ & + \cdots + \frac{1}{(k-1)!} B_{k-1}^{(k)}(h)\underset{\omega}{\Delta}^k \int_0^x \chi(x_1)\,dx_1 \\ & + \frac{1}{k!} B_k^{(k)}(h)\underset{\omega}{\Delta}^k \chi(x) + \cdots + \frac{1}{m!} B_m^{(k)}(h)\underset{\omega}{\Delta}^k \frac{d^{m-k}}{dx^{m-k}}\chi(x).\end{aligned} \tag{18}$$

Secondly we let $P=\Delta$, $Q=d^k/dx^k$, $P^{-1}=\sum_0^{x-1}\equiv\sum]_x-\sum]_0$ where $\sum=\Delta^{-1}$;

$$\begin{aligned}\chi(x+h) = {} & \frac{d^k}{dx^k} \sum_{x_1=0}^{x-1} \sum_{x_2=0}^{x_1-1} \cdots \sum_{x_k=0}^{x_{k-1}-1} \chi(x_k) \\ & + b_1^{(k)}(h)\,\frac{d^k}{dx^k} \sum_{x_1=0}^{x-1} \sum_{x_2=0}^{x_1-1} \cdots \sum_{x_{k-1}=0}^{x_{k-2}-1} \chi(x_{k-1}) + \cdots \\ & + \frac{1}{(k-1)!} b_{k-1}^{(k)}(h)\,\frac{d^k}{dx^k} \sum_0^{x-1} \chi(x_1) \\ & + \frac{1}{k!} b_k^{(k)}(h)\,\frac{d^k}{dx^k}\chi(x) + \cdots \\ & + \frac{1}{m!} b_m^{(k)}(h)\,\frac{d^k}{dx^k}\Delta^{m-k}\chi(x).\end{aligned} \tag{19}$$

In the above formulas $B_j^{(k)}(h)$ is the Bernoulli polynomial of order k and degree j, similarly $b_j^{(k)}(h)$ is the analogous polynomial which we designate as the Bernoulli polynomial of the second kind of order k and degree j.

[11] Nörlund, p. 29.

Formula (18) should be compared with a related formula of Nörlund.[12] If the operator $\sum_{x_1=0}^{x-1}\sum_{x_2=0}^{x_1-1}\cdots\sum_{x_k=0}^{x_{k-1}-1}$ is applied to both sides of (18) we get an interesting result which should be compared with a formula studied by the author in an earlier paper.[13]

4. **A remainder formula.** Let us assume that $\psi(x)$ is no longer a polynomial but that it is such a function that all operations applied to it in the sequel are meaningful. We consider formula (16) as our fundamental form.

We assume that $Q\psi(x)+\sum_{v=0}^{m}(F_v(t)/v!)P^vQ\psi(x)$ involves $m+1$ points, dependent upon x alone, which we call set 1. Some or all of these may coincide. For example, $\Delta^mQ\psi(x)$ involves $m+1$ distinct points and $(d^m/dx^m)Q\psi(x)$ $m+1$ coincident points. Let x be a constant and t a variable. When t is such that $x+t$ coincides with any one of the points of set 1

$$P^k\psi(x+t)-\left[Q\psi(x)+\sum_{v=1}^{m}\frac{F_v(t)}{v!}P^vQ\psi(x)\right]$$

vanishes, inasmuch as every term is identical with the like term which is built for the polynomial of degree m which coincides with $\psi(x)$ at the points of set 1. It has been remarked that (16) is an identity for a polynomial. Now denote by $p(t)$ a polynomial of degree $m+1$ which vanishes when $x+t$ coincides with any one of the $m+1$ points of set 1. Let the coefficient of t^{m+1} in $p(t)$ be 1. Choose a constant T such that

$$P^k\psi(x+t)-\left[Q\psi(x)+\sum_{v=1}^{m}\frac{F_v(t)}{v!}P^vQ\psi(x)\right]-Tp(t)$$

vanishes at the additional point where $t=h$. Then by repeated application of Rolle's theorem $(m+1)!T=d^{m+1}(P^k\psi(\xi))/dx^{m+1}$ where ξ is somewhere between the extreme values of the $(m+2)$ points composed of the points of set 1 and the additional point $x+h$. We have a final form for the remainder to formula (16),

$$R=\frac{1}{(m+1)!}p(h)\frac{d^{m+1}}{dx^{m+1}}(P^k\psi(\xi)). \tag{20}$$

The corresponding formula for (17) is

$$R=\frac{1}{(m+1)!}p(h)\frac{d^{m+1}}{dx^{m+1}}\chi(\xi). \tag{21}$$

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[12] Loc. cit., p. 160.
[13] Loc. cit.

FACTORIZATION OF DIFFERENTIAL IDEALS

HENRY J. RIBLET

1. Introduction. With the ultimate objective of introducing number theoretical considerations into the theory of algebraic differential fields,[1] conditions on a differential ring $\mathfrak{R}$ are given which are sufficient to insure that any differential ideal in it may be uniquely expressed as the product of a finite number of prime differential ideals. This program is justified by its intrinsic interest and the fact that the existence of such conditions suggests the possibility of obtaining a number theoretical complement for the theory of differential ideals in the ring of differential forms as developed by Ritt, Raudenbush and others,[2] paralleling the interdependent, classical theories of polynomial and number ideals.

With an additional axiom and certain alterations of the definitions, the proof of the basic theorems follows the pattern used by van der Waerden[3] in presenting the method of Krull.[4] Our procedure will be to discuss in detail only those questions raised by the altering of definitions.

2. Definitions. By a *differential ring* $\mathfrak{R}$, we shall mean a commutative ring with a unit element but no divisors of zero which is closed under differentiation. Its quotient field will be denoted by $\mathfrak{K}$. A *differential ideal* $\mathfrak{a}$ in $\mathfrak{R}$ is an ordinary ideal closed with respect to differentiation. $\mathfrak{R}$ will satisfy the *ascending chain condition* for differential ideals if every sequence of differential ideals, each properly containing the previous one, is finite in length. A differential ideal is said to be *divisorless* if the only differential ideal which properly contains it is the unit ideal $\mathfrak{R}$. A *principal* differential ideal is a differential ideal which is principal in the usual sense. The unit ideal is an example. $\mathfrak{R}$ will be said to be *integrally closed* if each element in $\mathfrak{K}$, having the property that all its integral powers have a fixed denominator, is in $\mathfrak{R}$. A differential ideal $\mathfrak{p}$ in $\mathfrak{R}$ will be called

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[1] H. J. Riblet, *Algebraic differential fields*, American Journal of Mathematics, vol. 53 (1941), pp. 339–446.

[2] J. F. Ritt, *Algebraic aspects of the theory of differential equations*, American Mathematical Society Semicentennial Publications, vol. 2, bibliography.

[3] B. L. van der Waerden, *Moderne Algebra*, vol. 2, p. 85.

[4] W. Krull, *Zur Theorie der allgemeinen Zahlringe*, Mathematische Annalen, vol. 99 (1928), pp. 51–70.

prime if the existence of two differential ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}\cdot\mathfrak{b}\equiv 0\ (\mathfrak{p})$ implies that $\mathfrak{a}\equiv 0\ (\mathfrak{p})$ or $\mathfrak{b}\equiv 0\ (\mathfrak{p})$.

It may be commented that the definition of an integrally closed differential ring is taken from Krull.[5] The more familiar definition[6] is a consequence of this plus the stronger assumption that an ascending chain condition holds for all ideals in $\mathfrak{R}$. It should be observed also that this definition of primeness for differential ideals is not equivalent to that of Raudenbush.[7] If we restrict ourselves to the perfect differential ideals[8] of $\mathfrak{R}$, however, the two definitions coincide.

3. **Axioms.** Let $\mathfrak{R}$ be a differential ring in which the following axioms are satisfied:

I. The ascending chain condition for differential ideals in $\mathfrak{R}$ holds.

II. Every prime differential ideal in $\mathfrak{R}$ is divisorless.

III. Every prime differential ideal in $\mathfrak{R}$ contains a principal differential ideal.

IV. $\mathfrak{R}$ is integrally closed.

With these axioms, we then prove the two basic theorems on prime factorization:

THEOREM 1. *Every differential ideal in $\mathfrak{R}$ is a product of a finite number of prime differential ideals.*

THEOREM 2. *If $\mathfrak{a}\equiv 0\ (\mathfrak{b})$, then each prime factor of $\mathfrak{a}$ always occurs at least as often as a prime factor of $\mathfrak{b}$.*

The proofs are identical with those given in Van der Waerden.[9] The only question occurs in using the symbol $\mathfrak{p}^{-1}$ to denote the set of elements in $\mathfrak{R}$ such that if a is one of them, $a\cdot\mathfrak{p}$ is in $\mathfrak{R}$. It is clear that $\mathfrak{p}^{-1}$ is closed with respect to addition and with respect to multiplication by elements of $\mathfrak{R}$. It is also true that it is closed with respect to differentiation for, if $a\in\mathfrak{p}^{-1}$ and $p\in\mathfrak{p}$, then $a\cdot p\in\mathfrak{R}$. Hence $a\cdot p^{(1)}-a^{(1)}\cdot p\in\mathfrak{R}$, where (1) denotes differentiation. Now $a\cdot p^{(1)}\in\mathfrak{R}$, so $a^{(1)}\cdot p\in\mathfrak{R}$; but p is any member of $\mathfrak{p}$ and accordingly $a^{(1)}\in\mathfrak{p}^{-1}$. Note that Axioms III and IV allow us to prove that $\mathfrak{p}\cdot\mathfrak{p}^{-1}=\mathfrak{R}$.

From the two theorems, it follows that divisibility implies factorization and that each differential ideal in $\mathfrak{R}$ may be expressed

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[5] W. Krull, ibid., p. 60.

[6] B. L. van der Waerden, loc. cit., p. 90.

[7] H. W. Raudenbush, Jr., *Ideal theory and algebraic differential equations*, Transactions of this Society, vol. 36 (1934), p. 365.

[8] H. W. Raudenbush, Jr., ibid., p. 362.

[9] B. L. van der Waerden, loc. cit., pp. 97–100.

uniquely as a product of a finite number of prime differential ideals. The consistency of the axioms is easily shown. If we define differentiation in the ring $C(x)$, obtained by adjoining x to the field of the rational numbers, in *any* way so as to leave it closed, it may be shown that Axioms I–IV are always satisfied. In $C(x, y)$ differentiation *may* be defined in such way that the statement as above still holds. This is of interest because every ordinary ideal in $C(x, y)$ may not be expressed uniquely as a product of a finite number of prime ideals.

HOFSTRA COLLEGE

ON THE ITERATION OF LINEAR HOMOGENEOUS TRANSFORMATIONS

ARNOLD DRESDEN

1. **Statement of problem.** The question which this note tries to answer is that of determining under what conditions on the matrix (a_{ij}), $(i, j=1, \cdots, n)$, the n-fold multiple sequence of complex numbers $x_k, x_k', \cdots, x_k^{(m)}, \cdots$ $(k=1, 2, \cdots, n)$ obtained by iteration of the linear homogeneous transformation $x_k' = a_{kj}x_j$ will converge for every initial set $x=(x_1, x_2, \cdots, x_n)$. Convergence is to be understood in the sense that there exists a set $X_1, X_2, \cdots, X_n$ such that, for $k=1, 2, \cdots, n$, $x_k^{(m)} \to X_k$, as $m \to \infty$.

2. **Jordan normal form.** We begin by recalling that a matrix $A=(a_{ij})$ with complex elements is similar to its Jordan normal form J_0. This means that there exists a unimodular matrix P, such that $A=P^{-1}J_0P$ and $J_0=PAP^{-1}$, where J_0 is the direct sum of Jordan matrices $J_1, \cdots, J_N$. To each elementary divisor $(\lambda-\lambda_\rho)^{e_\rho}$ of the characteristic matrix $\lambda I-A$ $(\rho=1, 2, \cdots, N)$ and $e_1+e_2+\cdots+e_N=n$, corresponds a Jordan matrix J_ρ. If $e_\rho>1$, then J_ρ has zero elements everywhere, except in the principal diagonal, all of whose elements are λ_ρ, and in the diagonal immediately below the principal diagonal, all of whose elements are 1.[1] If $e_\rho=1$, then J_ρ consists of the single element λ_ρ.

It follows that any integral power of J_0 is the direct sum of the same powers of the Jordan matrices J_ρ. Let us now denote by J an arbitrary Jordan matrix of order $n>1$,

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[1] See, for example, MacDuffee, *Introduction to Abstract Algebra*, p. 241.

$$J = \begin{pmatrix} \lambda & 0 & 0 & 0 \cdots & \cdot & 0 \\ 1 & \lambda & 0 & 0 \cdots & \cdot & 0 \\ 0 & 1 & \lambda & 0 \cdots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdots 0 & \lambda & 0 \\ 0 & \cdot & \cdot & \cdots 0 & 1 & \lambda \end{pmatrix}.$$

It is then easily seen that J^2 is a matrix of order n, whose first column is $\lambda^2, 2\lambda, 1, 0, \cdots, 0$, and whose second column is obtained from the first column by shifting the elements one place downward, as far as the length permits, and placing 0 in the first place; the third column is obtained from the second in similar manner; the last column is $0\ 0 \cdots 0\ \lambda^2$. A simple induction enables us to prove that, for any positive integer h, the matrix J^h is of order n, that its first column consists of the successive terms in the binomial expansion of $(\lambda+1)^h$, followed by zeros, provided $n \geqq h+1$ and otherwise of as many of these terms as the length of the column permits, that each of its columns after the first is obtained from the one to the left of it by shifting the elements one place downward and filling the open position above with a zero, and that its last column is $0\ 0\ 0 \cdots 0\ \lambda^h$. If J is a Jordan matrix of order 1, consisting of the single element λ then J^n consists of the single element λ^n.

3. **Reduction of problem; solution.** Returning now to our linear homogeneous transformation, we write it in the form

$$x' = xA, \tag{1}$$

where x and x' stand for the row-vectors $(x_1, x_2, \cdots, x_n)$ and $(x_1', x_2', \cdots, x_n')$, respectively. By means of the unimodular matrix P, we map the space of the row-vectors x on the space of the row-vectors y: $x = yP$, $\quad y = xP^{-1}$. This mapping carries the transformation (1) into a transformation in the y-space:

$$y' = x'P^{-1} = xAP^{-1} = yPAP^{-1} = yJ_0. \tag{2}$$

It follows from the mapping that the vectors $x^{(m)}$ will converge, in the sense described in §1, if and only if the vectors $y^{(m)}$ converge. But $y^{(m)} = y^{(m-1)}J_0 = \cdots = y_0J_0^m = y(J_1^m + J_2^m + \cdots + J_N^m)$. We obtain therefore the following form for the mth iterate of the transformation (2) in terms of the roots λ_ρ of the characteristic equation $|a_{ij} - \lambda\delta_{ij}| = 0$ and the exponents e_ρ of the corresponding elementary divisors $(\lambda - \lambda_\rho)^{e_\rho}$ of $\lambda I - A$, provided every $e_\rho > 1$.

$$y_1^{(m)} = \lambda_1^m y_1 + C_{m,1}\lambda_1^{m-1} y_2 + \cdots + C_{m,e_1-1}\lambda_1^{m-e_1+1} y_{e_1},$$

$$y_2^{(m)} = \lambda_1^m y_2 + C_{m,1}\lambda_1^{m-1} y_3 + \cdots + C_{m,e_1-2}\lambda_1^{m-e_1+2} y_{e_1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$y_{e_1}^{(m)} = \lambda_1^m y_{e_1},$$

$$y_{e_1+1}^{(m)} = \lambda_2^m y_{e_1+1} + C_{m,1}\lambda_2^{m-1} y_{e_1+2} + \cdots + C_{m,e_2-1}\lambda_2^{m-e_2+1} y_{e_1+e_2},$$

$$y_{e_1+1}^{(m)} = \lambda_2^m y_{e_1+2} + \cdots + C_{m,e_2-2}\lambda_2^{m-e_2+2} y_{e_1+e_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$y_{e_1+e_2}^{(m)} = \lambda_2^m y_{e_1+e_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$y_n^{(m)} = \lambda_N^m y_n.$$

This form makes it evident that the set $y_k^{(m)}$ will converge as m increases indefinitely, if and only if $|\lambda_\rho| < 1$, for $\rho = 1, \cdots, N$. If, on the other hand, all the elementary divisors involving a root λ_ρ are linear, then convergence will occur also if $\lambda_\rho = 1$. We have therefore obtained the following answer to our question:

THEOREM. *The iteration of the linear homogeneous transformation $x_k' = a_{kj}x_j$ $(k, j = 1, 2, \cdots, n)$ will converge for every initial set x_k if and only if all the roots of the characteristic equation of the matrix (a_{ij}) are less than unity in absolute value, except that the roots, which are involved only in linear elementary divisors, may be equal to unity.*[2]

4. **Remark.** It is clear that it is possible for some of the variables $y_k^{(m)}$ to converge to limits while others do not converge; for example, the iteration of the transformation $y_1' = 1/2y_1$, $y_2' = y_2 + y_3$, $y_3' = y_3 + y_4$, $y_4' = y_4$ will obviously yield a convergent sequence for $y_1^{(m)}$ and for $y_4^{(m)}$, but not for $y_2^{(m)}$, if we start with the initial set $y_1 = a$, $y_3 = b$, $y_2 = y_4 = 0$. In this case the roots of the characteristic equation are 1/2 and 1, with multiplicities 1 and 3, respectively. It may be worth while to determine, in the general case of the linear homogeneous transformation, which variables do and which do not converge upon iteration, in case some, but not all the roots of the characteristic roots of the matrix of the transformation are less than unity in absolute value.

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[2] In an earlier incomplete formulation of this theorem, the exceptional character of the linear elementary divisors was overlooked. The author is indebted for the detection of this error to Professor W. T. Reid, who called attention to the fact that convergence would take place if A were an idempotent matrix. In this connection it may be of interest to observe that for a matrix which satisfies an equation of the form $A^{r+1} = A^r$, $r \geqq 1$, *all* the elementary divisors of $\lambda I - A$ are linear. For, on the one hand, its Jordan normal form would have to satisfy the same relation, while it follows from our discussion that this can not be the case if $\lambda I - A$ has a non-linear elementary divisor.

A SUFFICIENT CONDITION FOR CESÀRO SUMMABILITY

G. D. NICHOLS

S. Chapman[1] has proved the summability (C, k) of the series $\sum r^s e^{r\theta}$ where $(k-1) \leqq s < k$, and $0 < \theta < 2\pi$. More recently M. S. MacPhail[2] has proved the exact summability (C, k) of the series $\sum P(r) \cdot f(r)$, where $P(r)$ is a polynomial of degree $k-1$ and $f(r)$ is a periodic function of mean value zero. He also gave a closed expression for the Cesàro "sum" of such a series. H. L. Garabedian[3] earlier published a special case of this result.

The purpose of the present paper is to prove the following closely related theorem in which the condition on the coefficients is somewhat more general. Also "sum" formulas are given for each of the sine and cosine series separately.

THEOREM. *The series* $(1/2)P(0) + \sum P(r) \cos rx$ *and* $\sum P(r) \sin rx$ *are summable* (C, k) *provided* $\{\Delta^k P(r)\}$ *is a monotone null sequence, and* $P(r)$ *together with its first* $k+1$ *derivatives each exist and are continuous for positive values of* r $(x \neq 2n\pi$ *in the cosine series).*

The proofs for the sine and cosine series are almost identical so only the proof for the latter will be given. Consider the series

$$S_n = (1/2)P(0) + \sum_{r=1}^{n} P(r) \cos rx. \tag{1}$$

If we multiply both sides of (1) successively k times by $2 \sin x/2$ there results, for k even,

$$(2^k \sin^k x/2)S_n = (1/2)D_{0k} + \sum_{r=1}^{(k-2)/2} D_{rk} \cos rx + (-1)^{k/2} \sum_{r=0}^{n-k} \Delta^k P(r) \cos (r + (k/2))\, x + (\mathrm{Cos})_k, \tag{2}$$

where the symbol $(\mathrm{Cos})_k$ represents k cosine terms, the highest order of any of the coefficients being the same as the order of $P(n)$.

For k odd, we have

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[1] S. Chapman, Proceedings of the London Mathematical Society, (2), vol. 9 (1911), p. 398.

[2] M. S. MacPhail, this Bulletin, vol. 47 (1941), p. 483.

[3] H. L. Garabedian, this Bulletin, vol. 45 (1939), p. 592.

$$(2^k \sin^k x/2)S_n = \sum_{r=1}^{(k-2)/2} C_{rk} \sin (2r-1)x/2 + (-1)^{(k+1)/2} \sum_{r=0}^{n-k} \Delta^k P(r) \sin (r+(k/2))x + (\mathrm{Sin})_k, \tag{3}$$

where $(\mathrm{Sin})_k$ has the same significance as $(\mathrm{Cos})_k$ above. For $k=1$ the first summation in (3) is to be omitted. The expressions for D_{rk} and C_{rk} in (2) and (3), respectively, are given in the sequel.

If we now apply the Cesàro definition for the "sum," σ, of a non-convergent series, there results from (2),

$$(2^k \sin^k x/2)\sigma = (1/2)D_{0k} + \sum_{r=1}^{(k-2)/2} D_{rk} \cos rx + \lim_{n\to\infty} (-1)^{k/2} \sum_{r=0}^{n-k} \frac{C(n-r+(k/2)+1, k)}{C(n+k, k)} \Delta^k P(r) \cos (r+(k/2))x + \lim_{n\to\infty} \frac{(\mathrm{Cos})_k}{(n+1)(n+2)\cdots(n+k)}, \tag{4}$$

where the C's following the second summation are the binomial coefficients.

To prove the existence of the first limit in (4) we first note that since there are the same number of factors in numerator and denominator of the fraction preceding $\Delta^k P(r)$, the fraction is positive and never greater than one for the values of n, r, and k under consideration. Since $\{\Delta^k P(r)\}$ is a monotone null sequence by hypothesis, the existence of the first limit (except for $x=2n\pi$) follows by Dirichlet's test.[4]

We wish to show that the second limit in (4) is equal to zero. Since the order of $(\mathrm{Cos})_k$ is the same as $P(n)$, this can be done by showing that if $\{\Delta^k P(n)\}$ is a monotone null sequence then the kth derivative of $P(n)$ approaches zero as n becomes infinite.

For this purpose we have

$$P^{(k)}(n) + (k/2)P^{(k+1)}(\xi) = \Delta^k P(n), \tag{5}$$

where $\xi \to \infty$ as $n \to \infty$. This is a special case of a formula due to Markoff,[5] and depends on the continuity of $P(n)$ and its first $k+1$ derivatives.

Since, by hypothesis, $\Delta^k P(n) \to 0$ as $n \to \infty$, it is only necessary that if $(y+(k/2)y') \to 0$ as $n \to \infty$, y and y' separately do so, where $y = P^{(k)}(n)$, and y and y' are continuous. A proof of this fact is given by Brom-

[4] Bromwich, *Theory of Infinite Series*, p. 49.

[5] Markoff, *Differenzenrechnung*, p. 21.

wich.[6] Hence the summability of the original series is established.

In case $P(r)$ is a polynomial of degree $k-1$, the first limit in (4) is also zero and σ is obtained in closed form from the first summation. Formulas for D_{rk} and C_{rk} are now given for this case. They can be established by induction, but the details are rather tedious and will be omitted

$$D_{rk} = \sum_{x=k/2+r-1}^{k-2} (-1)^{x+k+r+1}\Delta^{k-1-x}P(0)\left[C(x, (k/2)+r-1) + C(x, (k/2)-r-1)\right]. \tag{6}$$

For k odd, and $P(r)$ a polynomial of degree $k-1$, there results from (3),

$$(2^k \sin^k x/2)\sigma = \sum_{r=1}^{(k-1)/2} C_{rk} \sin (2r-1)x/2, \tag{7}$$

$$C_{rk} = \sum_{x=(k-2r-1)/2}^{k-2} (-1)^{x+k+r}\Delta^{k-1-x}P(0)\left[C(x, (k/2) + r - (3/2)) + C(x, (k/2)-r-(1/2))\right]. \tag{8}$$

For the sine series, $\sum P(r) \sin rx$, the results for $P(r)$, a polynomial of degree $k-1$, follow. For k odd,

$$(2^k \sin^k x/2)\sigma = \sum_{r=1}^{(k-1)/2} A_{rk} \cos (2r-1)x/2, \tag{9}$$

and for k even,

$$(2^k \sin^k x/2)\sigma = \sum_{r=1}^{(k-2)/2} B_{rk} \sin rx, \tag{10}$$

where A_{rk} and B_{rk} are given by

$$A_{rk} = \sum_{x=(k-2r-1)/2}^{k-2} (-1)^{x+k+r-1}\Delta^{k-1-x}P(0)\left[C(x, (k/2)-r-(1/2)) - C(x, (k/2)+r-(3/2))\right], \tag{11}$$

$$B_{rk} = \sum_{x=(k-2r-2)/2}^{k-2} (-1)^{x+k+r-1}\Delta^{k-1-x}P(0)\left[C(x, (k/2)-r-1) - C(x, (k/2)+r-1)\right]. \tag{12}$$

If the upper limit on $\sum$ in (4), (7), (9), and (10) is less than the lower limit, the value of the summation is to be taken as zero. Also $C(n, 0)=1$ for all n including zero, and $C(n, m)=0$ for $n<m$.

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[6] Bromwich, *Theory of Infinite Series*, p. 272.

ON THE THEORY OF THE TETRAHEDRON

N. A. COURT

I. DEFINITION. *We associate with the general tetrahedron* $(T) = ABCD$ *a sphere* (Q) *whose center is the Monge point* M *of* (T) *and the square of whose radius is*

$$q^2 = (MO^2 - R^2)/3, \tag{a}$$

where O *and* R *are the center and the radius of the circumsphere* (O) *of* (T).

In what follows, a number of propositions regarding the sphere (Q) will be established and it will be shown that from the properties of (Q) may be derived, as special cases, properties of the polar sphere (H) of the orthocentric tetrahedron (T_h).

For want of a better name we shall refer to (Q) as the "quasi-polar" sphere of the general tetrahedron (T).

The expression $MO^2 - R^2$ is the power of the Monge point M of (T) for the sphere (O).

THEOREM 1. *The square of the radius of the quasi-polar sphere of the general tetrahedron is equal to one-third of the power of the Monge point of the tetrahedron for its circumsphere.*

The sphere (Q) is real, a point sphere, or imaginary according as MO is greater than, equal to, or smaller than R. Moreover, we have $MO < 2R$, for the mid-point of MO is the centroid G of (T), and G necessarily lies within the sphere (O).

COROLLARY. *In an orthocentric tetrahedron* (T_h) *the Monge point coincides with the orthocenter* H, *and the above properties of* (Q) *are valid for the polar sphere* (H) *of* (T_h).[1]

The Monge point M of (T) is a center of similitude of the circumsphere (O) and the twelve point sphere (L) of (T),[2] hence M is the center of a sphere of antisimilitude of (O) and (L), that is, a sphere with respect to which the spheres (O) and (L) are inverse of one another.

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[1] Nathan Altshiller-Court, *Modern Pure Solid Geometry*, New York, 1935, p. 265, §813. This book will be referred to as MPSG.

[2] MPSG, p. 251, §764.

The line of centers LO of the spheres (L), (O) meets (O) in the points whose distances from M are $MO-R$ and $MO+R$. Again, the line LO meets (L) in two points whose distances from M are $ML-R/3$ and $ML+R/3$, for the radius of (L) is equal to $R/3$. Now the points $MO+R$, $ML-R/3$ correspond to each other in the inversion considered (and so do the points $MO-R$, $ML+R/3$), hence if x^2 is the constant of inversion, we have

$$x^2 = (MO + R)(ML - R/3) = (MO + R)(3ML - R)/3$$
$$= (MO^2 - R^2)/3 = q^2.$$

But the constant of inversion is the square of the radius of the sphere of inversion, and the sphere of inversion is coaxial with the two inverse spheres.

THEOREM 2. *The quasi-polar sphere of the general tetrahedron is coaxial with the circumsphere and the general twelve point sphere of the tetrahedron.*

COROLLARY. *To the square (L) of (T) corresponds in the orthocentric tetrahedron (T_h) the second twelve point sphere, and from the above property of (Q) we obtain the known property of the polar sphere of (T_h).*[3]

THEOREM 3. *The sum of the powers of the vertices of a general tetrahedron with respect to the quasi-polar sphere of that tetrahedron is equal to one-third of the sum of the squares of the edges of the tetrahedron.*

The power D_q of the vertex D of (T) for the sphere (Q) is

$$D_q = MD^2 - q^2 = MD^2 - (MO^2 - R^2)/3,$$

and from the triangle DMO we have

$$DM^2 + DO^2 = 2DG^2 + MO^2/2 = 2DG^2 + 2OG^2,$$

hence

$$D_q = 2DG^2 + 2(OG^2 - R^2)/3.$$

Adding to D_q the analogous formulas relative to the vertices A, B, C of (T) we obtain

$$A_q + B_q + C_q + D_q = 2(AG^2 + BG^2 + CG^2 + DG^2) + 8(OG^2 - R^2)/3.$$

Now if we denote by k^2 the sum of the squares of the edges of (T), we have

[3] MPSG, p. 264, §805.

$$AG^2 + EG^2 + CG^2 + DG^2 = k^2/4,^4$$

(b) $$OG^2 - R^2 = -k^2/16,^5$$

hence

$$A_q + B_q + C_q + D_q = k^2/2 - k^2/6 = k^2/3.$$

COROLLARY. *In an orthocentric tetrahedron the sum of the squares of a pair of opposite edges is constant, hence the sum of the powers of the vertices of the tetrahedron for the polar sphere is equal to the sum of the squares of a pair of opposite edges.*[6]

THEOREM 4. *The sum of the squares of the distances of the Monge point of a tetrahedron from the mid-points of the edges is equal to three times the sum of the squares of the radii of the circumsphere and the quasi-polar sphere of the tetrahedron.*

Let E, F be the mid-points of the edges DA, BC of (T). The mid-point of the bimedian $EF=m_a$ of (T) is the centroid G and therefore we have, from the triangle MEF,

$$ME^2 + MF^2 = 2MG^2 + EF^2/2 = 2OG^2 + m_a^2/2.$$

We have two analogous formulas relative to the other two pairs of opposite edges of (T). Adding the three relations we have

$$\sum ME^2 = 6OG^2 + (m_a^2 + m_b^2 + m_c^2)/2.$$

Now the expression in the parenthesis is equal to[7] $k^2/4$, hence, making use of the formula (b), we have

$$\sum ME^2 = 6(R^2 - k^2/16) + k^2/8 = 3R^2 + 3(R^2 - k^2/12).$$

But, eliminating OG between the formulas (a) and (b) we have

(c) $$R^2 - k^2/12 = q^2,$$

hence

$$\sum ME^2 = 3(R^2 + q^2).$$

COROLLARY. *The proposition is valid for the orthocentric tetrahedron.*[8]

THEOREM 5. *The sum of the powers of the Monge point of a tetra-*

[4] National Mathematics Magazine, vol. 15 (1941), p. 273, §3.
[5] V. Thebault, Nouvelles Annales de Mathématiques, (4), vol. 19 (1919), p. 425.
[6] Mathesis, vol. 42 (1928), p. 338, §2.
[7] MPSG, p. 56, §186.
[8] MPSG, p. 275, Example 4.

hedron with respect to the six spheres having for diameters the edges of the tetrahedron is equal to six times the square of the radius of the quasi-polar sphere of the tetrahedron.

The power of the Monge point M for the sphere (BC) having for diameter the edge $BC=a$ of (T) is equal to $MF^2-a^2/4$. We have five analogous expressions relative to the other five edges of (T). Adding the six expressions we have

$$\sum MF^2 - k^2/4,$$

or, using the value of $\sum MF^2$ from the preceding article,

$$6(R^2 - k^2/16) + k^2/8 - k^2/4 = 6q^2.$$

In an orthocentric tetrahedron the spheres having for diameters the edges of the tetrahedron are orthogonal to the polar sphere.

COROLLARY. *The power of the orthocenter for the six spheres is equal to the square of the radius of the polar sphere.*

The powers of the Monge point of (T) for the two spheres having for diameters a pair of opposite edges of (T) are equal.[9]

THEOREM 6. *The sum of the powers of the Monge point of a tetrahedron with respect to three spheres having for diameters three concurrent (or coplanar) edges of the tetrahedron is equal to three times the square of the radius of the quasi-polar sphere.*

The centroid G of (T) bisects the segment AA' joining A to the corresponding vertex A' of the tetrahedron (T') twin to (T), hence $AMA'O$ is a parallelogram.

Let A_1 be the diametric opposite of A on the circumsphere (O) of (T). The quadrilateral $MA'A_1O$ is a parallelogram, for OA_1 and MA' are equal and parallel. Hence the diagonal MA_1 bisects the diagonal OA', and therefore meets the median $A'G$ of the triangle $A'OM$ in the centroid G_a of that triangle. Thus

$$GG_a = A'G_a/2 = AG/3,$$

and G_a is therefore the centroid of the face BCD of (T). Moreover $MG_a=MA_1/3$.

Let A_2 be the second point of intersection, besides A, of the line AM with the sphere (O). The line A_1A_2 is perpendicular to the line AMA_2, hence if F is the foot of the perpendicular from G_a upon AMA_2 we have

[9] G. Gallucci, Nouvelles Annales de Mathématiques, (3), vol. 16 (1897), p. 17, §5.

$$MF:MA_2 = MG_a:MA_1 = 1:3,$$

and therefore

$$MA \cdot MA_2 = 3MA \cdot MF.$$

But $MA \cdot MA_2$ is the power of M for the sphere (O), hence by Theorem 1, $MA \cdot MF$ is equal to the square of the radius of the quasi-polar sphere (Q), that is, the points A, F are inverse with respect to (Q).

The median AG_a of (T) subtends a right angle at F, hence the sphere (AG_a) having AG_a for diameter is orthogonal to (Q), similarly for the three analogous spheres (BG_b), (CG_c), (DG_d). Thus this theorem follows:

THEOREM 7. *The quasi-polar sphere of a tetrahedron is orthogonal to the four spheres having for diameters the four medians of the tetrahedron.*[10]

Since the points A, F are inverse for (Q) (Theorem 7), the polar plane α of A for (Q) is perpendicular to AM at F; hence the line FG_a is the trace of α in the plane $AMGO$.

THEOREM 8. *The polar reciprocal tetrahedron of a given tetrahedron (T) with respect to the quasi-polar sphere of (T) is circumscribed about the medial tetrahedron of (T).*

THEOREM 9. *The faces of the polar reciprocal tetrahedron of a given tetrahedron (T) with respect to the quasi-polar sphere of (T) cut the spheres having for diameters the corresponding medians of (T) along four circles lying on the same sphere, namely, the twelve point sphere of (T).*

The point G_a lies on the twelve point sphere (L) of (T), and the diametric opposite of G_a on (L) lies on the line AM;[11] hence the point F also belongs to the sphere (L). Thus the plane α (Theorem 8) cuts the two spheres (AG_a) and (L) along the same circle, and FG_a is a diameter of that circle.

THEOREM 10. *The faces of the polar reciprocal tetrahedron of the medial tetrahedron of a given tetrahedron (T) with respect to the quasi-polar sphere of (T) cuts the spheres having for diameters the respective medians of (T) along four circles lying on the same sphere, namely the circumsphere of (T).*

The polar plane of G_a with respect to the quasi-polar sphere (Q) passes through A and is perpendicular to MG_a, hence the trace K

[10] American Mathematical Monthly, vol. 39 (1932), pp. 196, 197, §§11, 13.

[11] MPSG, p. 251.

of this plane on MG_a lies on the sphere (AG_a) having for diameter the median AG_a of (T).

On the other hand, the line MG_a passes through the diametric opposite A_1 of A on the circumsphere (O) of (T), hence K lies on (O). Thus the line AK is a common diameter of the two circles along which the polar plane considered cuts the two spheres (AG_a) and (O).

II. DEFINITIONS. *We associate with the general tetrahedron (T) the sphere having for center the centroid G of (T) and for the square of its diameter one-third of the sum of the squares of the bimedians of (T). The sphere will be denoted by (G) and referred to as the G-sphere of (T). The sphere (G) is concentric with the three spheres having for diameters the bimedians of (T).*

In an orthocentric tetrahedron the bimedians are equal, so that the three spheres having these bimedians for diameters coincide, and the G-sphere coincides with them, in the first twelve point sphere of the orthocentric tetrahedron, the three bimedians being diameters of that sphere.

THEOREM 1. *The quasi-polar sphere and the (G)-sphere of the general tetrahedron are orthogonal.*

The square of the radius, g^2, of the sphere (G) is equal to[7] (§I, Theorem 4)

$$(m_a^2 + m_b^2 + m_c^2)/3{:}4 = k^2/48,$$

hence

$$q^2 + g^2 = (R^2 - k^2/12) + k^2/48 = R^2 - k^2/16 = OG^2 = MG^2,$$

that is, the square of the line of centers of the spheres (Q), (G) is equal to the sum of the squares of their radii.

COROLLARY. *The polar sphere and the first twelve point sphere of the orthocentric tetrahedron are orthogonal.*[12]

THEOREM 2. *The sphere (G) belongs to the coaxial pencil formed by the circumsphere, the twelve point sphere, and the quasi-polar sphere of the tetrahedron.*

The centroid G is the second center of similitude, besides the point M, of the circumsphere (O) and the twelve point sphere (L) of (T). Now the quasi-polar sphere (Q) being a sphere of antisimilitude of

[12] MPSG, p. 262, §799.

(O) and (L) (§I, Theorem 2), the sphere (G) having G for center and orthogonal to (Q) (§II, Theorem 1) is the second sphere of anti-similitude of (O) and (L); hence (G) is coaxial with these spheres.

THEOREM 3. *The four spheres having for centers the vertices of a tetrahedron and orthogonal to the quasi-polar sphere cut the spheres having for diameters the respective medians of the tetrahedron along four circles belonging to the same sphere, namely, the (G)-sphere of the tetrahedron.*

The sphere (A) having A for center and orthogonal to the sphere (Q) is coaxial with the spheres (G) and (AG_a), for the centers of these three spheres are collinear and all three are orthogonal to (Q). Similarly for the vertices B, C, D of (T).

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EUCLIDEAN CONCOMITANTS OF THE TERNARY CUBIC

T. L. WADE

1. **Introduction; construction of concomitants.** In this paper we use the results of Cramlet [1] and the writer [2] to study the euclidean concomitants of the ternary cubic curve

$$T_{abc}X^aX^bX^c = 0,$$

where $a, b, c = 1, 2, 3$ and T_{abc} is symmetric. With tensor algebra as the medium of investigation all types of concomitants are readily constructed, and their geometric interpretations are also readily made in most cases. As is conventional in classical invariant theory, the word concomitant will be used as meaning rational integral concomitant unless stated to the contrary.

As a consequence of Theorem 3 in [2], we have the following theorem.

THEOREM I. *Every euclidean concomitant of the ground form $T_{abc}X^aX^bX^c$ $(a, b, c = 1, 2, 3)$ is expressible by composition as a tensor of order zero with the use of the coefficient tensor T_{abc}, the variable coordinate tensors X^a and U_a, and the numerical tensors ϵ^{abc}, L_a, and E^{ab}.*

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The tensor ϵ_{abc} will not appear here since we are dealing with a ground form in X^a. In detail,

$$L_a = [0, 0, 1], \quad E^{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is convenient to introduce the tensor A^{ab}, where

$$A^{ab} = \epsilon^{abc}L_c = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 3 of [2] is the first fundamental theorem of euclidean invariant theory in tensor form, and Theorem I above is the particular form of it which we need here; this theorem constitutes a basis for the construction of concomitants.

2. **Reduction of concomitants.** In [2] an algebraically complete system of euclidean invariants for the cubic $T_{abc}X^aX^bX^c = 0$ is given in tensor form. For the conic $C_{ab}X^dX^b = 0$ the familiar algebraically complete system of three invariants is also an irreducibly complete system. But this is not true for the cubic. In this paper we shall find an irreducible system, complete through the fifth degree, of euclidean invariants for the cubic, and shall investigate geometric interpretation of these invariants in connection with other concomitants of the ground form. It is generally recognized that the problem of reduction of concomitants is more difficult than the problem of construction. See H. Weyl [3].

It is of interest to note that between the simple case of the conic and the situation for the general cubic is that of the degenerate cubic considered as the line $B_aX^a = 0$ and the conic $C_{ab}X^aX^b = 0$. They have the irreducibly complete system of eight invariants

$$I_1 = B_aB_bE^{ab}, \qquad I_2 = C_{ab}C_{cd}C_{ef}\epsilon^{ace}\epsilon^{bdf}, \qquad I_3 = C_{ab}C_{cd}A^{ac}A^{bd},$$
$$I_4 = C_{ab}E^{ab}, \qquad I_5 = C_{ab}E^{ac}E^{bd}B_cB_d, \qquad I_6 = C_{ac}C_{bd}\epsilon^{abe}\epsilon^{cdf}B_eB_f,$$
$$I_7 = C_{ab}C_{cd}\epsilon^{ace}A^{bd}B_e, \qquad I_8 = C_{ab}A^{ac}E^{bd}B_cB_d.$$

Six of these are algebraically independent; they may be chosen as $I_1, \cdots, I_6$; then I_7 and I_8 are expressible in terms of them by means of the syzygies

$$3(I_7)^2 \equiv 3I_3I_6 - 2I_1I_2I_4 - 2I_2I_5,$$
$$(I_8)^2 \equiv I_1I_4I_5 - (I_5)^2 - \tfrac{1}{2}(I_1)^2I_3.$$

The five invariants I_2, I_3, I_6, I_7, and $C_{ab}A^{ac}A^{bd}B_cB_d$ form an irreducibly complete system of affine invariants for the line and conic. They were studied recently by Weitzenböck [4], who used the symbolic notation of his text [5].

We now state the second fundamental theorem in the detailed and explicit tensor form needed to investigate the relations on the concomitants constructed in accordance with the first fundamental theorem.

THEOREM II. *Every identity satisfied by the concomitants constructed by the first fundamental theorem for a set of ground forms in X^a can be established with the basic identities:*

Iden 1. $\epsilon^{abc}E^{de} \equiv \epsilon^{dbc}E^{ae}+\epsilon^{adc}E^{be}+\epsilon^{abd}E^{ce}$,

Iden 2. $\epsilon^{abc}X^d \equiv \epsilon^{dbc}X^a+\epsilon^{adc}X^b+\epsilon^{abd}X^c$,

Iden 3. $\epsilon^{abc}\epsilon^{def} \equiv \epsilon^{dbc}\epsilon^{aef}+\epsilon^{adc}\epsilon^{bef}+\epsilon^{abd}\epsilon^{cef}$,

Iden 4. $A^{ab}A^{cd} \equiv E^{ac}E^{bd}-E^{ad}E^{bc}$,

Iden 4′. $E^{ac}E^{bd} \equiv A^{ab}A^{cd}+E^{ad}E^{bc}$,

Iden 5. $A^{ab}E^{cd} \equiv A^{cb}E^{ad}+A^{ac}E^{bd}$,

Iden 6. $A^{ab}X^c \equiv A^{cb}X^a+A^{ac}X^b+\epsilon^{abc}L_dX^d$,

Iden 7. $A^{ab}\epsilon^{cde} \equiv A^{cb}\epsilon^{ade}+A^{ac}\epsilon^{bde}+\epsilon^{abc}A^{de}$,

Iden 8. $\epsilon^{abc}A^{de} \equiv \epsilon^{dbc}A^{ae}+\epsilon^{adc}A^{be}+\epsilon^{abd}A^{ce}$,

Iden 9. $A^{ab}A^{cd} \equiv A^{cb}A^{ad}+A^{ac}A^{bd}$,

Iden 10. $E^{ab}E^{cd}E^{ef} \equiv E^{ab}E^{cf}E^{ed}+E^{ad}E^{cb}E^{ef}-E^{ad}E^{cf}E^{eb}-E^{ef}E^{cb}E^{ed}+E^{ad}E^{cf}E^{eb}$,

Iden 11. $X^aE^{bc}E^{de} \equiv X^aE^{be}E^{dc}+E^{ac}X^bE^{de}-E^{ac}E^{be}X^d-E^{ae}X^bE^{dc}+E^{ae}E^{bc}X^d+\epsilon^{abd}A^{ec}L_fX^f$.

Iden 2 may be established by expanding

$$\begin{vmatrix} \delta_1^a & \delta_2^a & \delta_3^a & X^a \\ \delta_1^b & \delta_2^b & \delta_3^b & X^b \\ \delta_1^e & \delta_2^e & \delta_3^e & X^e \\ \delta_1^d & \delta_2^d & \delta_3^d & X^d \end{vmatrix} \equiv 0,$$

and using

$$\epsilon^{abc} = \begin{vmatrix} \delta_1^a & \delta_2^a & \delta_3^a \\ \delta_1^b & \delta_2^b & \delta_3^b \\ \delta_1^e & \delta_2^e & \delta_3^e \end{vmatrix}.$$

Most of the other identities are consequences of Iden 2 in rather obvious ways. We remark briefly about some not of this class. To obtain Iden 4 consider

$$A^{ab}A^{cd} \equiv \epsilon^{ab\beta}\epsilon^{cd\beta} = \begin{vmatrix} \delta_1^a & \delta_2^a \\ \delta_1^b & \delta_2^b \end{vmatrix} \cdot \begin{vmatrix} \delta_1^c & \delta_1^d \\ \delta_2^c & \delta_2^d \end{vmatrix} \equiv \begin{vmatrix} E^{ac} & E^{ad} \\ E^{bc} & E^{bd} \end{vmatrix}.$$

Iden 10 may be obtained from the determinantal product

$$\begin{vmatrix} \delta_1^a & \delta_2^a & \delta_3^a \\ \delta_1^b & \delta_2^b & \delta_3^b \\ \delta_1^c & \delta_2^c & \delta_3^c \end{vmatrix} \cdot \begin{vmatrix} \delta_1^d & \delta_1^e & \delta_1^f \\ \delta_2^d & \delta_2^e & \delta_2^f \\ 0 & 0 & 0 \end{vmatrix} \equiv 0,$$

and Iden 11 from

$$\begin{vmatrix} \delta_1^a & \delta_2^a & \delta_3^a \\ \delta_1^b & \delta_2^b & \delta_3^b \\ \delta_1^c & \delta_2^c & \delta_3^c \end{vmatrix} \cdot \begin{vmatrix} \delta_1^d & \delta_1^e & X^1 \\ \delta_2^d & \delta_2^e & X^2 \\ 0 & 0 & X^3 \end{vmatrix} \equiv \epsilon^{abc}A^{de}L_fX^f.$$

That these identities constitute a reduction basis for the concomitants constructed by the first fundamental theorem may be seen by observing that they give the alternate ways of writing all types of products which arise.

3. **An irreducible system, complete through the fifth degree, of euclidean invariants for the cubic** $T_{abc}X^aX^bX^c$. We note that the cubic has only one "formal" invariant of the first degree, $T_{abc}\epsilon^{abc}$, and this vanishes identically due to the skew-symmetry of ϵ^{abc}; so (i) for the cubic there is no invariant of the first degree.

We can construct three (and only three) invariants of the second degree:

$$A = T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1a_2}E^{b_1b_2}E^{a_3b_3}, \qquad B = T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1b_1}E^{a_2b_2}E^{a_3b_3},$$

$$C = T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1b_1}A^{a_2b_2}E^{a_3b_3}.$$

Invariants as $T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1b_1}A^{a_2b_2}A^{a_3b_3}$ and $T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1a_2}E^{b_1b_2}E^{a_3b_3}$ which vanish identically on the interchange of equivalent indices are not listed here, nor in similar circumstances in the future. In this connection one should keep in mind the skew-symmetry of ϵ^{abc} and of A^{ab}. The only identities applicable to A and B are Idens 4′ and 10. The application of the latter to each of these merely gives $A \equiv A$ and $B \equiv B$. The application of Iden 4′ to either of them results in $C \equiv A - B$. The only identities applicable to C are Idens 4 and 5. The first results in the relation just given, and the latter in $C \equiv C$. Thus $C \equiv A - B$ is the only relation on the invariants A, B, C. Therefore, (ii) for the cubic there are two irreducible invariants of the second degree, and these may be chosen as

$$II_1 = T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1a_2}E^{b_1b_2}E^{a_3b_3}, \qquad II_2 = T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1b_1}E^{a_2b_2}E^{a_3b_3}.$$

Often we shall use the contracted notation

$$_{(k)}T = T_{a_{11}a_{12}a_{13}}T_{a_{21}a_{22}a_{23}} \cdots T_{a_{k1}a_{k2}a_{k3}}.$$

To illustrate, $II_1 = {}_{(2)}T \cdot E^{a_1a_2}E^{b_1b_2}E^{a_3b_3}$.

There are three invariants of the third degree to consider:

$$A = {}_{(3)}T \cdot A^{a_1b_1}E^{b_2c_1}A^{c_2a_2}\epsilon^{a_3b_3c_3}, \qquad B = {}_{(3)}T \cdot E^{a_1c_1}E^{b_1b_2}A^{c_2a_2}\epsilon^{a_3b_3c_3},$$

$$C = {}_{(3)}T \cdot A^{a_1b_1}A^{b_2c_1}A^{c_2a_2}\epsilon^{a_3b_3c_3}.$$

Applying Iden 5 to $A^{c_2a_2}E^{b_1b_2}$ of B, we get $B = -A - A$, or $B = -2A$. Applying Iden 4 to $A^{a_1b_1}A^{b_2c_1}$ of C, we obtain $C = A - B$. On application of the appropriate reduction formulas in all possible ways to A, B, C, it is found that there is no relation on these expressions other than the two given. Hence: (iii) For the cubic there is one irreducible invariant of the third degree, and this may be chosen as

$$III = (1/6)T_{a_1a_2a_3}T_{b_1b_2b_3}T_{c_1c_2c_3}A^{a_1b_1}A^{b_2c_1}A^{a_2c_2}\epsilon^{a_3b_3c_3}.$$

The number of formal invariants which one can construct increases rapidly as the degree goes beyond three. Their consideration is facilitated by considering all invariants of a certain type together. (iv-1) For the cubic there is only one invariant of the fourth degree which contains the numerical tensor ϵ^{abc} four times, and this is irreducible, it being the well known projective invariant of the fourth degree

$$IV_1 = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2d_1}\epsilon^{a_3c_2d_2}\epsilon^{b_2c_3d_3}.$$

There are six invariants of the fourth degree which contain factors like ϵ^{abc} twice, the remaining factors being like E^{ab}:

$$A = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{a_3d_1}E^{b_3d_2}E^{c_3d_3}, \quad B = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{a_3b_3}E^{c_3d_1}E^{d_2d_3},$$

$$C = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2a_3}E^{b_3c_3}E^{d_2d_3}, \quad D = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2a_3}E^{b_3d_2}E^{c_3d_3},$$

$$E = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2d_2}E^{a_3d_3}E^{b_3c_3}, \quad F = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2b_3}E^{c_3d_2}E^{a_3d_3}.$$

Applying the basic identities in all possible ways, we find that these invariants are connected by the relations

$$A \equiv D + 2F; \quad B \equiv E + 2F; \quad B \equiv C + 2D; \quad C \equiv 2D + E - 2F;$$

and only these. Therefore: (iv-2) For the cubic there are two irreducible invariants of the fourth degree which contain factors like ϵ^{abc} twice, the remaining factors being like E^{ab}, and these may be chosen as

$$IV_2 = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2a_3}E^{b_1c_3}E^{d_2d_3},$$

$$IV_3 = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2b_3}E^{c_3d_2}E^{a_3d_3}.$$

Like considerations lead us to the conclusions: (iv-3) For the cubic there is one irreducible invariant of the fourth degree which contains factors like ϵ^{abc} twice, the other factors being like E^{ab}, and this may be chosen as

$$IV_4 = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{b_3c_3}E^{d_1d_2}A^{a_3d_3}.$$

(iv-4) For the cubic there is one irreducible invariant of the fourth degree whose numerical tensor product is composed entirely of factors like E^{ab}, except for one factor like A^{ab}, and this may be chosen as

$$IV_5 = {}_{(4)}T \cdot E^{a_1a_2}A^{a_3b_1}E^{b_2c_2}E^{b_3c_1}E^{c_3d_1}E^{d_2d_3}.$$

For invariants of the fourth degree constructed wholly with the aid of E^{ab}, there are five possibilities:

$$A = {}_{(4)}T \cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_1}E^{b_3c_2}E^{c_3d_1}E^{d_2d_3},$$

$$B = {}_{(4)}T \cdot E^{a_1b_1}E^{a_2b_2}E^{a_3c_1}E^{b_3d_1}E^{c_2d_2}E^{c_3d_3},$$

$$C = {}_{(4)}T \cdot E^{a_1b_1}E^{a_2c_1}E^{a_3d_1}E^{b_2c_2}E^{b_3d_2}E^{c_3d_3},$$

$$D = {}_{(4)}T \cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_1}E^{b_3d_1}E^{c_2d_2}E^{c_3d_3},$$

$$E = {}_{(4)}T \cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_1}E^{b_3d_1}E^{c_2c_3}E^{d_2d_3}.$$

Applying the identities 1–11 in all possible ways it is found that there are four and only four relations on these expressions; so: (iv-5) For the cubic there is only one irreducible invariant constructed wholly with the aid of E^{ab}, of the fourth degree, and this may be chosen as

$$IV_6 = {}_{(4)}T \cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_1}E^{b_3c_2}E^{c_3d_1}E^{d_2d_3}.$$

Then

$$B \equiv IV_6 + (1/2)(II_2)^2 - (1/2)(II_1)^2,$$

$$C \equiv IV_6 + II_1 \cdot II_2 - (II_1)^2,$$

$$D \equiv IV_6 + (1/2)II_1 \cdot II_2 - (1/2)(II_1)^2,$$

$$E \equiv IV_6 + (1/2)(II_1)^2 - (1/2)II_1 \cdot II_2.$$

All other invariants of the fourth degree are found to be reducible; so we may summarize as follows: (iv) For the cubic there are six irreducible invariants of the fourth degree, and these may be chosen as $IV_1, IV_2, IV_3, IV_4, IV_5, IV_6$.

In a similar manner it is found that: (v) For the cubic there are two irreducible invariants of the fifth degree, and these may be chosen as

$$V_1 = {}_{(5)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}\epsilon^{a_3d_1c_1}A^{d_2c_2}E^{b_2c_3}E^{d_1c_3},$$

$$V_2 = {}_{(5)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}\epsilon^{a_3d_1c_1}A^{b_1c_2}E^{d_2d_3}A^{c_1c_3}.$$

A typical reduction is that of $A = {}_{(5)}T\cdot\epsilon^{d_1b_1c_1}A^{a_1b_2}E^{a_2d_3}E^{c_2c_3}E^{b_3d_2}E^{a_3c_1}E^{c_1c_3}$. Iden 5 applied to $A^{a_1b_2}E^{d_2a_3}$ of A gives $A \equiv 4II_1\cdot III - A$, or $A \equiv 2II_1\cdot III$.

We may combine (i), (ii), (iii), (iv), (v) in the following theorem.

THEOREM III. *For the cubic $T_{abc}X^aX^bX^c$ ($a, b, c = 1, 2, 3$) there are eleven irreducible euclidean invariants of degree less than six, and these may be chosen as II_1, II_2, III, IV_1, IV_2, IV_3, IV_4, IV_5, IV_6, V_1, V_2.*

4. **An irreducible system, complete through the fourth degree, of euclidean covariants for the cubic.** There seems to have been no systematic study of euclidean covariants for the cubic curve of the third order. The best known are the Hessian, the polar conic of the line at infinity, and the Laplacian. We shall often speak of a covariant of degree i in T_{abc} and order j in X^a as a (i, j) covariant. The line at infinity is a (0, 1) covariant, having for its equation $\Omega = L_aX^a = X^3 = 0$.

4.1. $(i, 1)$ *covariants.* For the cubic $T_{abc}X^aX^bX^c$ there is only one (1, 1) covariant, and this is $L_1 = T_{abc}E^{ab}X^c$. There is no (2, 1) covariant. There are four irreducible (3, 1) covariants, and these may be chosen as

$$L_{31} = {}_{(3)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{b_2c_3}X^{a_3}, \qquad L_{32} = {}_{(3)}T\cdot E^{a_1b_1}E^{a_2c_1}E^{c_2c_3}E^{b_2b_3}X^{a_3},$$

$$L_{33} = {}_{(3)}T\cdot A^{a_1b_1}E^{a_2c_1}E^{b_2c_2}E^{b_3c_3}X^{a_3}, \qquad L_{34} = {}_{(3)}T\cdot A^{a_1b_1}E^{a_2c_1}E^{b_2b_3}E^{c_1c_3}X^{a_3}.$$

The other (3, 1) covariants

$$A = {}_{(3)}T\cdot E^{a_1b_1}E^{a_2c_1}E^{b_2c_2}E^{b_3c_3}X^{a_3}, \qquad B = {}_{(3)}T\cdot E^{a_1b_1}E^{a_2b_2}E^{b_3c_1}E^{c_2c_3}X^{a_3},$$

$$C = {}_{(3)}T\cdot A^{a_1b_1}E^{a_2b_2}E^{b_3c_1}E^{c_2c_3}X^{a_3}, \qquad D = {}_{(3)}T\cdot E^{a_1b_1}E^{a_2b_3}A^{b_2c_1}E^{c_2c_3}X^{a_3},$$

are expressible in terms of the irreducible ones by the relations:

$$A \equiv L_{32} + (II_2 - II_1) + 2III\cdot\Omega,$$

$$B \equiv L_{32} + (1/2)(II_2 - II_1)L_1 + 2III\cdot\Omega,$$

$$C \equiv L_{33},$$

$$D \equiv L_{34} - L_{33}.$$

In like manner it is found that there is only one irreducible (4, 1) covariant, and this may be chosen as

$$L_4 \equiv {}_{(4)}T\cdot\epsilon^{a_1b_1c_1}E^{a_2b_2}E^{d_1d_2}E^{b_3c_2}E^{c_3d_3}X^{a_3}.$$

Therefore we have this theorem.

THEOREM IV. *For the cubic there are seven irreducible* $(i, 1)$ *euclidean covariants* $(i=0, 1, 2, 3, 4)$, *and these may be chosen as* $\mathfrak{L}$, L_1, L_{31}, L_{32}, L_{33}, L_{34}, *and* L_4.

4.2. $(i, 2)$ *covariants*. Obviously there is no (1, 2) covariant. There are four (2, 2) covariants:

$$A = {}_{(2)}T\cdot E^{a_1b_1}E^{a_2b_2}X^{a_3}X^{b_3}, \qquad B = {}_{(2)}T\cdot A^{a_1b_1}A^{a_2b_2}X^{a_3}X^{b_3},$$
$$C = {}_{(2)}T\cdot A^{a_1b_1}E^{b_2b_3}X^{a_2}X^{a_3}, \qquad D = {}_{(2)}T\cdot E^{a_1b_1}E^{b_2b_3}X^{a_2}X^{a_3}.$$

The expressions C and D contain no significant reducible factor, and consequently are irreducible. Iden 4 applied to $A^{a_1b_1}A^{a_2b_2}$ gives $B\equiv(L_1)^2-A$. This is the only relation that exists on A and B; so (i) there are three irreducible (2, 2) covariants, and these may be chosen as

$$C_{21} = {}_{(2)}T\cdot A^{a_1b_1}A^{a_2b_2}X^{a_3}X^{b_3}, \qquad C_{22} = {}_{(2)}T\cdot A^{a_1b_1}E^{b_2b_3}X^{a_2}X^{a_3},$$
$$C_{23} = {}_{(2)}T\cdot E^{a_1b_1}E^{b_2b_3}X^{a_2}X^{a_3}.$$

One may construct three (3, 2) covariants

$$A = {}_{(3)}T\cdot\epsilon^{a_1b_1c_1}A^{a_2b_2}E^{c_2c_3}X^{a_3}X^{b_3}, \qquad B = {}_{(3)}T\cdot\epsilon^{a_1b_1c_1}A^{a_2c_2}E^{b_2c_3}X^{a_3}X^{b_3},$$
$$C = {}_{(3)}T\cdot\epsilon^{a_1b_1c_1}A^{b_2c_2}E^{b_3c_3}X^{a_2}X^{a_3}.$$

Applying Iden 5 to $A^{a_2b_2}C^{c_2c_3}$ of A we obtain $A\equiv 2B$. Iden 6 applied to $A^{a_2c_2}X^{b_3}$ of B gives $B\equiv C-B-L_{31}\cdot\mathfrak{L}$, or $2B\equiv C-L_{31}\cdot\mathfrak{L}$. There is no other relation on these expressions. Therefore (ii) there is one irreducible (3, 2) covariant, and this may be chosen as

$$C_3 = {}_{(3)}T\cdot\epsilon^{a_1b_1c_1}A^{b_2c_2}E^{b_3c_3}X^{a_2}X^{a_3}.$$

(iii) By a similar procedure it is found that there are five irreducible (4, 2) covariants, and these may be chosen as

$$C_{41} = {}_{(4)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{c_3d_1}E^{d_2d_3}X^{a_3}X^{b_3},$$
$$C_{42} = {}_{(4)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2d_1}E^{c_2c_3}E^{d_2d_3}X^{a_3}X^{b_3},$$
$$C_{43} = {}_{(4)}T\cdot\epsilon^{b_1c_1d_1}\epsilon^{b_2c_2d_2}E^{a_1b_3}E^{c_2d_3}X^{a_2}X^{a_3},$$
$$C_{44} = {}_{(4)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{d_1d_2}A^{c_3d_3}X^{a_3}X^{b_3},$$
$$C_{45} = {}_{(4)}T\cdot\epsilon^{b_1c_1d_1}\epsilon^{b_2c_2d_2}E^{c_3d_3}A^{b_3a_1}X^{a_2}X^{a_3}.$$

Combining (i), (ii), and (iii) we have this theorem

THEOREM V. *For the cubic there are nine irreducible* $(i, 2)$ *euclidean covariants* $(i=1, 2, 3, 4)$, *and these may be chosen as* C_{21}, C_{22}, C_{23}, C_3, C_{41}, C_{42}, C_{43}, C_{44}, C_{45}.

Using the general methods of construction and reduction as we have above, we find that this theorem follows.

THEOREM VI. *For the cubic* $(i=1, 2, 3, 4)$: (i) *there are six euclidean irreducible* $(i, 3)$ *covariants, and these may be chosen as*

$$T_{31} = {}_{(3)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}X^{a_2}X^{b_2}X^{c_2},$$
$$T_{32} = {}_{(3)}T\cdot A^{a_1b_1}E^{b_2c_1}E^{c_2c_3}X^{b_2}X^{a_2}X^{a_3},$$
$$T_{33} = {}_{(3)}T\cdot A^{b_1c_1}E^{b_2c_2}E^{a_1c_3}X^{b_2}X^{a_2}X^{a_3},$$
$$T_{41} = {}_{(4)}T\cdot\epsilon^{a_1b_1c_1}E^{b_2d_1}E^{c_2d_2}E^{c_3d_3}X^{b_2}X^{a_2}X^{a_3},$$
$$T_{42} = {}_{(4)}T\cdot\epsilon^{d_1b_1c_1}E^{b_2c_2}E^{a_1d_2}E^{c_3d_3}X^{b_2}X^{a_2}X^{a_3},$$
$$T = T_{abc}X^aX^bX^c.$$

(ii) *There are three irreducible* $(i, 4)$ *covariants, and these may be chosen as*

$$Q_2 = {}_{(2)}T\cdot E^{a_1b_1}X^{a_2}X^{a_3}X^{b_2}X^{b_3},\quad Q_{41} = {}_{(4)}T\cdot\epsilon^{b_1c_1d_1}\epsilon^{b_2c_2d_2}E^{a_1d_2}X^{a_2}X^{b_2}X^{c_3},$$
$$Q_{42} = {}_{(4)}T\cdot\epsilon^{b_1c_1d_1}\epsilon^{b_2c_2d_2}A^{a_1d_2}X^{a_2}X^{a_3}X^{b_2}X^{c_3}.$$

(iii) *There are two irreducible quintics, and these may be chosen as*

$$R_3 = {}_{(3)}T\cdot E^{a_1c_1}A^{b_1c_2}X^{a_2}X^{a_3}X^{b_2}X^{b_3}X^{c_3},$$
$$R_4 = {}_{(4)}T\cdot\epsilon^{a_1c_1d_1}E^{b_1c_2}E^{d_2d_3}X^{a_2}X^{a_3}X^{b_2}X^{b_3}X^{c_3}.$$

(iv) *There is no irreducible covariant for* $i<5$, $j>6$.

5. **Geometric interpretations.** It should be understood that the general cubic curve given by the general equation of the third degree in X^a,

$$T = T_{abc}X^aX^bX^c = 0,$$

$(a, b, c=1, 2, 3$ and T_{abc} symmetric) is under consideration here. For such cubic no invariant is zero, and no covariant vanishing identically, unless specifically stated so. The point $P_2=T_{a_1a_2a_3}T_{b_1b_2b_3}\cdot\epsilon^{a_1b_1r}A^{a_2b_2}E^{a_3b_3}U_r=0$ in expanded form is

$$(T_{112}T_{123}+T_{122}T_{223}-T_{122}T_{113}-T_{123}T_{222})U_1$$
$$(T_{122}T_{123}+T_{112}T_{113}-T_{112}T_{223}-T_{111}T_{123})U_2$$
$$(T_{111}T_{112}-(T_{112})^2-(T_{122})^2+T_{112}T_{222})U_3=0.$$

Keeping this expanded form in mind, and examining the contents of the paper by Thomae [6], or that by Stuyvaert [7], we observe the truth of this theorem.

THEOREM VII. P_2 *is the unique point whose polar conic is a circle.*

When we speak of the polar of a point (or of the pole of a line), unless specified to the contrary, we mean with respect to the fundamental cubic T.

THEOREM VIII. (a) *The polar conic of P_2 is C_3, and* (b) *C_3 is a circle.*

Obviously the polar conic of Y^a with respect to T has the equation $T_{abc}Y^aX^bX^c=0$. Let $Y^rU_r=T_{a_1a_2a_3}T_{b_1b_2b_3}\epsilon^{a_1b_1r}A^{a_2b_2}E^{a_3b_3}U_r$. Then (a) is evident, and (b) follows.

THEOREM IX. *Every covariant conic of the cubic T whose coefficients are of the third degree in T_{abc} is a circle.*

In §4.2 it was shown that one can construct three (3, 2) covariants, there designated by A, B, C, and that these are connected by the relations $A\equiv 2B$, and $2B\equiv C-L_{21}\cdot\mathfrak{L}$. By Theorem VIII, C_3 is a circle, and as a consequence of the relations just given A and B are also circles. Recall from §4 that there are no covariant conics of degree less than two, that there are covariant lines of the first and third degree, but none of the second, and further from §3 there is no invariant of degree less than two. As a consequence of these facts we conclude that every covariant conic Q of the third degree is of the form $Q\equiv k_1C_3+k_2R$, where k_1 and k_2 are constants, and R is at most linear in X^a. But since C_3 is a circle, every such conic Q is a circle.

The condition that the polar conic of the point Y^a, $T_{abc}X^aX^bY^c$, be a rectangular hyperbola is that $(T_{11c}+T_{22c})Y^c=T_{abc}E^{ab}Y^c=0$.

THEOREM X. *The line $L_1=0$ is the locus of points whose polar conics are rectangular hyperbolas.*

This line is called the Laplacian of the cubic. The cubic for which $L_1\equiv 0$ has been studied by Brooks [8].

THEOREM X′. *The Laplacian of the Hessian $H=(3)T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}X^{a_1}X^{b_2}X^{c_3}=0$ is $L_{21}=0$.*

Adapting the argument of White [9] to tensor notation, we find that the equation of the polar conic of the line $V_aX^a=0$ with respect to T is

$$T_{a_1a_2a_3}T_{b_1b_2b_3}\epsilon^{a_1b_1p}\epsilon^{a_2b_2q}X^{a_3}X^{b_3}U_pU_q=0.$$

If in this we replace U_p by $L_p=[0, 0, 1]$ and use $A^{ab}=\epsilon^{abc}L_c$, we have this next theorem.

THEOREM XI. *The polar conic of the line at infinity is $C_{21}=0$.*

In like manner, we have this theorem.

THEOREM XI′. *The polar conic of the Laplacian is* $C_{42}=0$.

The polar line of the point $Y^rU_r=0$ with respect to $C_{21}=0$ is

$$T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1b_1}A^{a_2b_2}X^{a_3}X^{a_3}Y^{b_3}=0.$$

In particular let $Y^rU_r=0$ be P_2. We have for the polar of P_2 with respect to C_{21}

$$\begin{aligned}K &= T_{a_1a_2a_3}T_{b_1b_2b_3}T_{c_1c_2c_3}T_{d_1d_2d_3}A^{a_1b_1}A^{a_2b_2}\epsilon^{c_1d_1b_1}A^{c_2d_2}E^{c_3d_3}X^{a_3}\\ &= 4III\cdot L_1-{}_{(4)}T\cdot\epsilon^{d_1b_1c_1}A^{d_2b_2}E^{d_3b_3}E^{a_2c_2}E^{a_3c_3}X^{a_1}\ \text{(by Iden 4)}.\end{aligned}$$

Using other identities we find that the last term is $2III\cdot L_1$; so $K\equiv 2III\cdot L_1$.

THEOREM XII. *The Laplacian of a cubic is the polar with respect to the polar conic of the line at infinity of the unique point whose polar conic is a circle.*

Rather evident are these theorems and corollary.

THEOREM XIII. *The polar conic of* $T_{abc}E^{ab}A^{cr}U_r=0$, *the point at infinity on the Laplacian, is* $C_{22}=0$.

COROLLARY. $C_{22}=0$ *is a rectangular hyperbola.*

THEOREM XIV. *The polar conic of* $T_{abc}E^{ab}E^{cr}U_r=0$, *the point at infinity in the direction perpendicular to the Laplacian, is* $C_{23}=0$.

Taking the polar of $T_{abc}E^{ab}E^{cr}U_r=0$ and $T_{abc}E^{ab}A^{cr}U_r=0$, respectively, with respect to $C_{23}=0$, we get these theorems.

THEOREM XV. *The diameter of the conic* C_{23} *conjugate to the direction perpendicular to the Laplacian is* $L_{32}=0$.

THEOREM XVI. *The diameter of the conic* C_{23} *conjugate to the direction of the Laplacian is* $L_{34}=0$.

The condition for two lines V_a and W_a to be perpendicular is that $V_aW_bE^{ab}=0$, and the condition for them to be parallel is that $V_aW_bA^{ab}=0$. From these a number of geometric facts follow quite directly:

A.1. The locus of points whose linear polars are parallel to the Laplacian is the conic $C_{22}=0$.

2. The locus of points whose linear polars are perpendicular to the Laplacian is the conic $C_{23}=0$.

3. The locus of points whose linear polars with respect to the Hessian are perpendicular to the Laplacian is the conic $C_{41}=0$.

4. The locus of points whose linear polars with respect to the Hessian are parallel to the Laplacian is the conic $C_{44}=0$.

5. The locus of points whose linear polars are perpendicular to the Laplacian of the Hessian is the conic $C_{43}=0$.

6. The locus of points whose linear polars are parallel to the Laplacian of the Hessian is the conic $C_{45}=0$.

7. The locus of points whose linear polars with respect to the fundamental cubic and the Hessian are perpendicular is the quartic $Q_{41}=0$.

8. The locus of points whose linear polars with respect to the fundamental cubic and the Hessian are parallel is the quartic $Q_{42}=0$.

9. The locus of points whose linear polars with respect to fundamental cubic and the quartic Q_2 are parallel is the quintic $R_3=0$.

10. The Laplacian of the fundamental cubic and the Laplacian of the Hessian are parallel if (and only if) $IV_4=0$, and they are perpendicular if

$$(4)\quad T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{b_1c_1}E^{a_2d_1}E^{d_2d_2}\equiv IV_2+2IV_3=0.$$

11. The Laplacian and $L_{32}=0$ are parallel if $IV_6=0$, and they are perpendicular if

$$(4)\quad T\cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_2}E^{b_3c_3}E^{c_1d_1}E^{d_2d_3}\equiv IV_6+(1/2)(II_1)^2-(1/2)II_1\cdot II_2=0.$$

The line $V_aX^a=0$ is said to be minimal if $V_aV_bE^{ab}=0$. Hence we have the following facts.

B.1. The Laplacian is a minimal line if (and only if) $II_1=0$.

2. The locus of points whose linear polars are minimal lines is the quartic $Q_2=0$.

Using the condition for incidence of the point V_a and the line Y^a, namely that $V_aY^a=0$, we get these statements.

C.1. P_2, the unique point whose polar conic is a circle, lies on the Laplacian if $III=0$.

2. P_2 lies on the Laplacian of the Hessian if $V_1=0$.

Not so direct is the next fact.

3. The line through P_2 and perpendicular to the Laplacian is $L_{33}=0$.

6. **Some concomitants in expanded form.** We list a few typical concomitants in expanded form. These are obtained by carrying out the indicated summations in the tensor-invariant forms of the concomitants, using the defined values of the components of ϵ^{abc}, A^{ab}, and E^{ab}.

$$II_1=T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1a_2}E^{b_1b_2}E^{a_3b_3}=(T_{111}+T_{122})^2+(T_{112}+T_{222})^2,$$

$$II_2=T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1b_1}E^{a_2b_2}E^{a_3b_3}$$

$$=(T_{111})^2+3(T_{112})^2+3(T_{122})^2+(T_{222})^2,$$

$$III = (1/6)T_{a_1a_2a_3}T_{b_1b_2b_3}T_{c_1c_2c_3}A^{a_1b_1}A^{b_2c_1}A^{a_2c_2}\epsilon^{a_3b_3c_3}$$

$$= \begin{vmatrix} T_{111} & T_{112} & T_{113} \\ T_{211} & T_{212} & T_{213} \\ T_{221} & T_{222} & T_{223} \end{vmatrix},$$

$$IV_5 = {}_{(4)}T \cdot E^{a_1a_2}A^{a_3b_1}E^{b_2c_1}E^{b_3c_2}E^{c_3d_1}E^{d_2d_3}$$

$$= 3(T_{111})^2T_{112}T_{123} + 2(T_{123})^3T_{222} + 3T_{111}(T_{123})^2T_{222}$$
$$+ 6T_{111}T_{112}(T_{123})^2 + T_{111}(T_{222})^3 + 3T_{112}(T_{123})^3$$
$$- 3T_{112}T_{123}(T_{222})^2 - 2T_{111}(T_{112})^3 - 3T_{111}(T_{112})^2T_{222}$$
$$- 6(T_{112})^2T_{123}T_{222} - (T_{111})^3T_{222} - 3(T_{112})^2T_{123},$$

$$L_1 = T_{abc}E^{ab}X^c = (T_{111} + T_{112})X^1 + (T_{112} + T_{222})X^2$$
$$+ (T_{113} + T_{223})X^3,$$

$$(1/2)C_{21} = T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1b_1}A^{a_2b_2}X^{a_3}X^{b_3}$$

$$= [T_{111}T_{123} - (T_{113})^2](X^1)^2 + [T_{111}T_{223} - T_{112}T_{123}]X^1X^2$$
$$+ [T_{112}T_{223} - (T_{223})^2](X^2)^2$$
$$+ [T_{123}T_{113} + T_{111}T_{223} - 2T_{112}T_{123}]X^1X^3$$
$$+ [T_{112}T_{223} + T_{113}T_{222} - 2T_{123}T_{123}]X^2X^3$$
$$+ [T_{113}T_{223} - (T_{123})^2](X^3)^2.$$

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GENERALIZATION OF A THEOREM OF KOROUS ON THE BOUNDS OF ORTHONORMAL POLYNOMIALS

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1. **Introduction.** An elementary treatment of the convergence of series of orthogonal polynomials is greatly facilitated if the polynomials of the orthonormal set are known to be uniformly bounded on the domain of orthogonality, or on a part of it where convergence is to be proved.[1] A demonstration due to J. Korous[2] shows in a few lines that the orthonormal polynomials corresponding to a weight function $\rho\sigma$ on a finite interval are thus bounded, if the polynomials for weight ρ have the desired property, and if the factor σ satisfies a Lipschitz condition and has a positive lower bound on the entire domain of orthogonality. The purpose of this note is to show that the argument of Korous can be extended so as to apply under fairly general conditions to orthogonal polynomials in two real variables on an algebraic curve[3] and in particular to orthogonal trigonometric sums,[4] which can be regarded as orthogonal polynomials on a circle.[5] A problem of the same category has been discussed by Peebles[6] with less simple hypotheses on the factor σ.

In the case of trigonometric sums it is known in advance that the orthonormal functions for weight $\rho \equiv 1$, namely $(2\pi)^{-1/2}$, $\pi^{-1/2} \cos kx$, $\pi^{-1/2} \sin kx$, $k = 1, 2, \cdots$, are uniformly bounded, and a theory of the convergence of developments in series of orthogonal trigonometric sums is opened up immediately. For other algebraic curves the question of the existence of a weight function which gives rise to a bounded system of orthonormal polynomials is one requiring separate investigation, and the answer to this question is known at present only in particular instances.[7] When the existence of a single such

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[1] See, for example, D. Jackson, *Series of orthogonal polynomials*, Annals of Mathematics, (2), vol. 34 (1933), pp. 527–545; pp. 531–538.

[2] See G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, p. 157; D. Jackson, *Fourier Series and Orthogonal Polynomials*, Carus Mathematical Monographs, no. 6, 1941, pp. 205–208.

[3] See D. Jackson, *Orthogonal polynomials on a plane curve*, Duke Mathematical Journal, vol. 3 (1937), pp. 228–236. This paper will be cited by the letter A.

[4] See, for example, D. Jackson, *Orthogonal trigonometric sums*, Annals of Mathematics, (2), vol. 34 (1933), pp. 799–814. This paper will be cited as B.

[5] [A, p. 234.]

[6] G. H. Peebles, this Bulletin, abstract 45-5-219.

[7] See, for example, D. Jackson, this Bulletin, abstract 45-5-192; Fulton Koehler,

ρ has been established for a specified domain of integration of the type to be discussed below, the conclusions of the present note lead to a wide generalization of the convergence theory for series of orthogonal polynomials associated with the curve in question.

2. **Statement of the problem.** Let C be an algebraic curve or a portion of an algebraic curve, degenerate or nondegenerate, consisting of a single connected piece of finite extent or a finite number of such pieces. Let C be of the mth degree, in the sense that there is a polynomial $\Pi(x, y)$ of the mth degree in x and y, but no polynomial of lower degree, which vanishes identically on C, while each irreducible factor of Π vanishes on a portion of C of positive length. The expression "connected pieces" will be understood to mean "pieces each of which is connected," not "pieces connected with each other." A single "piece," on the other hand, may be made up of parts corresponding to different irreducible factors of Π; it may be, for example, the perimeter of a square, or a set of line segments radiating from a common point. Let $\rho(x, y)$ be a non-negative weight function which is integrable over C with respect to arc length, and for each irreducible factor of Π is positive on a set of positive measure on the corresponding part of C.

Under these circumstances[8] there is a set of polynomials in x and y orthonormal on C with arc length as variable of integration and ρ as weight function, which includes just m polynomials of the nth degree for each value of $n \geqq m$, and $n+1$ polynomials of the nth degree for $n < m$. Let $p_{ni}(x, y)$, $n = 0, 1, 2, \cdots$; $i = 1, 2, \cdots, m$, be such an orthonormal set, with the supplementary convention that when $n+1 < m$, $p_{ni} \equiv 0$ for $i > n+1$, and also, on occasion, that $p_{-1,i} \equiv 0$ for $1 \leqq i \leqq m$. The value of $\int_C \rho p_{ni}^2 ds$ then is in every case either 1 or 0.

Let $\sigma(x, y)$ be a function defined and having a positive lower bound h on C, and satisfying the condition that if P_1 and P_2 are any two points on a connected portion of C, with coordinates (x_1, y_1) and (x_2, y_2),

$$| \sigma(x_2, y_2) - \sigma(x_1, y_1) | \leqq \lambda \Delta s,$$

where λ is constant (for simplicity of notation, a single constant for the whole of C), and Δs is the distance from P_1 to P_2 along C, or the shortest such distance, if P_1 and P_2 are connected by C in more than one way.

Systems of orthogonal polynomials on certain algebraic curves, this Bulletin, vol. 46 (1940), pp. 345–351.

[8] See [A, pp. 232–234]. The present hypotheses are somewhat more general than those which were explicitly formulated in the earlier paper, but no essential change is required in the reasoning which leads to recognition of the form of the orthonormal system.

Let $q_{ni}(x, y)$, $n=0, 1, 2, \cdots; i=1, 2, \cdots, n+1$ for $n<m$, $i=1, 2, \cdots, m$ for $n \geqq m$, be a set of orthonormal polynomials on C with $\rho\sigma$ as weight function; there is no occasion to introduce identically vanishing q's.

The question at issue is that of inferring boundedness of the polynomials q_{ni} from that of the polynomials p_{ni}. It will be found that such an inference is possible, under conditions to be specified.

3. **Proof of the theorem for an arbitrary algebraic curve.** Let

$$K_n(x, y, u, v) = \sum_{k=0}^{n} \sum_{i=1}^{m} p_{ki}(x, y) p_{ki}(u, v).$$

Then any one of the polynomials $q_{nj}(x, y)$, as a polynomial of the nth degree, has on C the representation

$$\begin{aligned} q_{nj}(x, y) &= \int_C \rho(u, v) K_n(x, y, u, v) q_{nj}(u, v) ds \\ &= \sum_{i=1}^{m} p_{ni}(x, y) \int_C \rho(u, v) p_{ni}(u, v) q_{nj}(u, v) ds \\ &\quad + \int_C \rho(u, v) K_{n-1}(x, y, u, v) q_{nj}(u, v) ds; \end{aligned} \tag{1}$$

the integration is extended over the curve C in the (u, v)-plane. By application of Schwarz's inequality when $p_{ni} \not\equiv 0$,

$$\begin{aligned} \left[\int_C \rho p_{ni} q_{nj} ds\right]^2 &\leqq \int_C \rho p_{ni}^2 ds \int_C \rho q_{nj}^2 ds = \int_C \rho q_{nj}^2 ds \\ &\leqq \frac{1}{h} \int_C \rho\sigma q_{nj}^2 ds = \frac{1}{h}, \end{aligned} \tag{2}$$

the p's being normalized for weight ρ and the q's for weight $\rho\sigma$. It remains to examine in some detail the integral involving K_{n-1}. Let this integral be denoted by $I(x, y)$ or I.

The sum $K_{n-1}(x, y, u, v)$ has alternative representations of the form[9]

$$K_{n-1} = \frac{1}{u - x} \sum_{i=1}^{m} \sum_{l=1}^{m} c_{il} [p_{ni}(u, v) p_{n-1,l}(x, y) - p_{n-1,l}(u, v) p_{ni}(x, y)], \tag{3}$$

$$K_{n-1} = \frac{1}{v - y} \sum_{i=1}^{m} \sum_{l=1}^{m} d_{il} [p_{ni}(u, v) p_{n-1,l}(x, y) - p_{n-1,l}(u, v) p_{ni}(x, y)], \tag{4}$$

[9] [A, pp. 235–236.]

as special cases of a more general expression having an arbitrary linear combination of $u-x$ and $v-y$ in the denominator. If G is a number such that $|x| \leqq G$, $|y| \leqq G$ on C, then

$$|c_{il}| \leqq G, \qquad |d_{il}| \leqq G$$

for all values of the subscripts. For

$$c_{il} = \int_C \rho u p_{ni} p_{n-1,l} ds,$$

and by another application of Schwarz's inequality

$$c_{il}^2 \leqq \int_C \rho u^2 p_{ni}^2 ds \int_C \rho p_{n-1,l}^2 ds \leqq G^2 \int_C \rho p_{ni}^2 ds = G^2;$$

a similar calculation applies to d_{il}.

Since $K_{n-1}(x, y, u, v)$ is for fixed (x, y) a polynomial in (u, v) of degree lower than the nth, it follows from the property of orthogonality of q_{nj} that

$$\int_C \rho(u, v)\sigma(u, v)K_{n-1}(x, y, u, v)q_{nj}(u, v)ds = 0.$$

Consequently $\sigma(x, y)I$ can be represented in the form

$$(5) \qquad \sigma(x, y)I = \int_C \rho(u, v)[\sigma(x, y) - \sigma(u, v)]K_{n-1}(x, y, u, v)q_{nj}(u, v)ds.$$

Let C_1 be a portion of C forming a closed point set, consisting of a finite number of connected pieces, and containing no singular point, that is, neither a singular point of any nondegenerate component of C nor any point common to two such components. Then each connected part of C_1 is a straight line segment, a smooth arc, or a smooth closed loop.

The essential ideas of the argument are illustrated with a simpler initial formulation, and in a form adequate for some of the most interesting applications, if it is assumed that C itself has the characteristics ascribed to C_1, and that C_1 is the whole of C; the method is however of wider applicability as indicated. On the other hand, the reasoning could be made still more general, at the expense of some further attention to details, but the usefulness of the additional generality would not be so immediately apparent. The need for explicit consideration of certain wholly elementary items of detail arises from the fact that for fixed (x, y) either of the denominators $u-x$, $v-y$

in the expressions for K_{n-1} may vanish not only at the point $u=x$, $v=y$, but also at a number of other points of the curve.

At any point of C_1 the derivatives dy/dx and dx/dy have definite values, or else one of them has the value 0. Except on straight line segments, dy/dx can take on the values ± 1 at only a finite number of points. The locus C_1 is made up of a finite number of arcs or segments, on any one of which $|dy/dx|$ is everywhere greater than or equal to 1 or else everywhere less than or equal to 1. Any such arc C' (the word *arc* being regarded now and henceforth as applicable in particular to a line segment) is included in or coextensive with an arc C'', also belonging to C, on which $|dy/dx|$ is everywhere greater than or equal to $1/2$, if it is greater than or equal to 1 on C', or everywhere less than or equal to 2, if it is less than or equal to 1 on C', and such that each end of C', if not a terminal point of C, is an interior point of C''. Let a definite arc C'' be associated in this way with each C'. The distance from any point of an arc C' to any point of C not belonging to the corresponding arc C'' has a positive lower bound δ for the whole of the locus C_1 which constitutes the aggregate of the arcs C'. It is to be noted further that $|ds/dx| \leqq 5^{1/2}$ on an arc C'' where $|dy/dx| \leqq 2$, and $|ds/dy| \leqq 5^{1/2}$ on the other type of arc C''.

Let P, with coordinates (x, y), be an arbitrary point of C_1. Let C_P' be the arc C' to which P belongs, or, if P is a common end point of two such arcs, let C_P' be either of them. Let K be the corresponding arc C''. Let the rest of C be subdivided into two parts K', K'', so that if P', with coordinates (u, v), is any point of K', the line PP' makes an angle not greater than $\pi/4$ with the x-axis, and if P' is on K'', PP' makes an angle less than $\pi/4$ with the y-axis. With the value of δ defined in the preceding paragraph, $|u-x| \geqq \delta/2^{1/2}$ for any point P' on K', and $|v-y| \geqq \delta/2^{1/2}$ if P' is on K''.

Let it be supposed for definiteness that C_P' is an arc of slope numerically less than or equal to 1; the alternative case would be treated in the same way, with the obvious interchanges of variables. For integration in (5), the point (x, y) being regarded as fixed, let $K_{n-1}(x, y, u, v)$ be represented by (3) when (u, v) is on K or K', and by (4) when (u, v) is on K''.

On K, since $\Delta s \leqq 5^{1/2}|\Delta x|$,

$$|\sigma(x, y) - \sigma(u, v)| \leqq 5^{1/2}\lambda |u - x|.$$

Let μ be an upper bound for σ on the whole of C. Then on K',

$$|[\sigma(x, y) - \sigma(u, v)]/(u - x)| \leqq 2^{3/2}\mu/\delta,$$

and there is a similar inequality on K'' with $u-x$ replaced by $v-y$.

By application of Schwarz's inequality, as in (2),

$$\int_C \rho(u, v) \left| p_{\alpha\beta}(u, v) q_{ni}(u, v) \right| ds \leqq k^{-1/2},$$

and the same upper bound is valid if the integration is extended only over K, K', or K''.

Let it be supposed that $|p_{\alpha\beta}(x, y)| \leqq H$ *on a point set* C_2 *contained in* C_1, *uniformly for all values of the subscripts* α *and* β. (In particular, C_2 may be the whole of C_1.) Then, by combination of the inequalities that have been obtained,

$$\sigma(x, y) \left| I(x, y) \right| \leqq (5^{1/2}\lambda + 2^{5/2}\mu/\delta) 2m^2 G k^{-1/2} H$$

if (x, y) is on C_2. (It is readily seen that the coefficient $2^{5/2} = 2 \cdot 2^{3/2}$ could be replaced by $2^{3/2}$, but the difference is immaterial for the purpose in hand.) The absolute value of the sum in the third member of (1) does not exceed $mk^{-1/2}H$. Consequently, for (x, y) on C_2,

$$\left| q_{ni}(x, y) \right| \leqq H[mk^{-1/2} + (5^{1/2}\lambda + 2^{5/2}\mu/\delta) 2m^2 G k^{-3/2}].$$

Since the right-hand member is independent of x, y, j, and n, it follows that *if the* p*'s are uniformly bounded on* C_2, *the same is true of the* q*'s.*

4. **Orthogonal trigonometric sums.** In a particular case the results can be interpreted as relating to orthogonal trigonometric sums, the unit circle or a finite set of arcs on the unit circle being taken for C and C_1. With $x = \cos\theta$, $y = \sin\theta$, the polynomials $p_{ni}(x, y)$, $q_{ni}(x, y)$ are trigonometric sums in θ. The weight function $\rho(x, y)$ and the factor $\sigma(x, y)$ will be represented alternatively by $\rho(\theta)$ and $\sigma(\theta)$. As far as the definition of the orthonormal system is concerned, the case of a domain of orthogonality consisting of detached intervals in a period is of course equivalent to that of a domain consisting of the entire period, with a weight function which vanishes identically outside the intervals in question. The conclusion can be stated as follows:

Let $u_0(\theta)$, $u_1(\theta)$, $v_1(\theta)$, $\cdots$ *constitute a set of orthonormal trigonometric sums with respect to a weight function* $\rho(\theta)$ *on a domain* D_1 *consisting of a period interval, regarded as closed, or of a finite number of closed intervals contained in a period; let* $\sigma(\theta)$ *have a positive lower bound on* D_1, *and satisfy a condition of the form*

$$\left| \sigma(\theta_2) - \sigma(\theta_1) \right| \leqq \lambda \left| \theta_2 - \theta_1 \right|$$

on each interval of D_1, *taking on the same value at both ends of the period interval if these end points belong to* D_1; *and let* $U_0(\theta)$, $U_1(\theta)$, $V_1(\theta)$, $\cdots$

be the orthonormal trigonometric sums on D_1 for weight $\rho(\theta)\sigma(\theta)$; if the u's and v's are uniformly bounded on a point set D_2 contained in D_1, the same is true of the U's and V's.

For this case the proof admits a materially simpler formulation than when geometric configurations are contemplated having the degree of generality previously considered. The details relating to the loci C', C'', K, K', K'', can be dispensed with for the most part; with θ replacing the pair of coordinates (x, y), and ϕ replacing the pair (u, v), it is sufficient, for any particular value of θ, to consider separately the intervals $(\theta-\pi/2, \theta+\pi/2)$ and $(\theta+\pi/2, \theta+3\pi/2)$, and in the integral corresponding to the right-hand member of (5) to represent $K_{n-1}(\theta, \phi)$ in the former interval by an expression with denominator $\sin(\theta-\phi)$, and in the latter interval by an alternative expression with $1-\cos(\theta-\phi)$ in the denominator.[10]

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[10] [B, pp. 808–809.]

APPROXIMATION OF CONTINUOUS FUNCTIONS BY MEANS OF LACUNARY POLYNOMIALS

BERNARD DIMSDALE

1. **Introduction.** All rational integral polynomials are linear combinations of members of the complete set of powers whose exponents are the non-negative integers. If certain members of this set are deleted, the linear combinations formed from the resulting set are, in the strict sense of the term, "lacunary polynomials." In a large part of this paper, however, methods of reasoning designed for the treatment of such polynomials are applicable to combinations from much more general sets of powers whose exponents are non-negative but not in general integral. The term "polynomial in x^μ of degree μ_n" will be applied to combinations from the set $1, x^{\mu_1}, x^{\mu_2}, \cdots$ where $\mu_1, \mu_2, \cdots$ form an arbitrarily preassigned set of real numbers such that $0<\mu_1<\mu_2<\cdots$, and μ_n is the largest exponent.

This paper started out as an investigation of lacunary orthogonal polynomials, and although this aspect of it became subordinate to the

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problem of approximation, the relation of the latter to the theory of convergence of series of the orthogonal polynomials is pointed out at the end.

2. **Preliminary results.** It has been shown[1] that for all positive integers m the trigonometric sum

$$I_m(t) = h_m \int_{-\pi/2}^{\pi/2} g(t + 2u)F_m(u)du$$

with

$$F_m(u) = \left[\frac{\sin mu}{m \sin u}\right]^4, \qquad \frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} F_m(u)du,$$

of order $2m-2$, satisfies $|I_m(t)-g(t)| \leqq D\lambda/m$, for all t, provided that $g(t)$ has period 2π and satisfies a Lipschitz condition, constant λ, for all real t, where D is an absolute constant; and, furthermore, that if $g(t)$ is an even function, $I_m(t)$ will be a cosine sum, $I_m(t) = \alpha_0^{(m)} + \alpha_1^{(m)} \cos t + \cdots + \alpha_{2m-2}^{(m)} \cos (2m-2)t$.

Now

$$\alpha_k^{(m)} = \frac{1}{\pi} \int_0^{2\pi} I_m(t) \cos kt\, dt,$$

and when $I_m(t)$ is replaced by its integral form, order of integration reversed, and the substitution $t+2u=y$ made,

$$\alpha_k^{(m)} = \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \left[\int_0^{2\pi} \cos k(y - 2u)g(y)dy \right] du,$$

since $g(t)$ has period 2π. Since[2]

$$\left| \frac{1}{\pi} \int_0^{2\pi} g(t) \begin{matrix}\cos \\ \sin\end{matrix} kt\, dt \right| \leqq \frac{\pi\lambda}{k}$$

it follows on expanding $\cos k(y-2u)$ that[3]

$$|\alpha_k^{(m)}| \leqq \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \frac{\pi^2\lambda}{k} du + \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \frac{\pi^2\lambda}{k} du,$$

$$|\alpha_k^{(m)}| \leqq \frac{2\pi\lambda}{k}, \qquad k = 1, 2, \cdots, m-1; m = 2, 3, \cdots.$$

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[1] D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. 11, 1930, pp. 2–6.

[2] See, for example, Titchmarsh, *The Theory of Functions*, p. 426.

[3] The following shorter proof has been given $|\alpha_k^{(m)}| = |(1/\pi)\int_0^{2\pi} [I_m(t)-g(t)+g(t)] \cos kt\, dt| \leqq 2D\lambda/m + \pi\lambda/k < \lambda(2D+\pi)/k$. The different value of the constant here would change subsequent computations slightly.

If $g(x)$ satisfies a Lipschitz condition, constant λ, on the closed interval[4] $[-1, 1]$, then for each positive integer m there exists a polynomial $P_m(y)$ in $1, y, y^2, \cdots$, of degree $2m-2$, such that on $[-1, 1]$

$$|g(y) - P_m(y)| \leqq \frac{D\lambda}{m}.$$

Here[5] $P_m(y) = I_m(t)$, where $I_m(t)$ is defined for $g(\cos t)$ and $y = \cos t$. Let $P_m(y) = a_0^{(m)} + a_1^{(m)} y + \cdots + a_{2m-2}^{(m)} y^{2m-2}$. The following inequalities result directly from the fact that $P_m(y) = I_m(t)$;

$$|a_{2k}^{(m)}| \leqq \frac{2^{2k}\pi\lambda}{2k} C_{2k}(m + k - 1),$$

$$|a_{2k-1}^{(m)}| \leqq \frac{2^{2k-1}\pi\lambda}{2k-1} C_{2k-1}(m + k - 2),$$

where

$$C_r(s) = \frac{s!}{(s-r)!r!}, \qquad k = 1, 2, \cdots, m-1; m = 2, 3, \cdots.$$

Let $y = x^H$, $g(x^H) = f(x)$, $H > 0$. Then[6] $|f(x_2) - f(x_1)| \leqq \lambda |x_2^H - x_1^H|$ on $[0, 1]$, implies that for every positive integer m there exists a polynomial

$$P_{mH}(x) = a_0^{(m)} + a_1^{(m)} x^H + \cdots + a_{2m-2}^{(m)} x^{(2m-2)H}$$

such that on $[0, 1]$,

$$|f(x) - P_{mH}(x)| \leqq \frac{D\lambda}{m}.$$

It will be observed that in this transformation the coefficients $a_k^{(m)}$ remain unchanged.

It has been shown by Szász[7] that for a sequence $\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n$, where $0 < \gamma_0$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n$ and $\gamma_0 \neq \gamma_i$, $i = 1, 2, \cdots, n$, there exists a set of constants $u_1, u_2, \cdots, u_n$ such that

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[4] Hereafter the bracket notation for an interval implies closure at both ends.

[5] See the argument in D. Jackson. loc. cit., pp. 13–14.

[6] Hereafter this condition will be called a "Lipschitz condition H." It has been pointed out that this condition implies that $f(x)$ satisfies a Lipschitz condition of order α, $\alpha = H$ if $H \leqq 1$, $\alpha = 1$ if $H \geqq 1$. However, the converse is not true, since $x = 0$ belongs to the interval in question. Hence this condition is more restrictive than a Lipschitz condition of order α. On any closed interval not including 0 the two conditions are, however, equivalent.

[7] O. Szász, *Über die Approximation Stetiger Funktionen durch lineare Aggregate von Potenzen*, Mathematische Annalen, vol. 77 (1916), pp. 485–487.

$$\int_0^1 [x^{\gamma_0-1/2} + u_1 x^{\gamma_1-1/2} + \cdots + u_n x^{\gamma_n-1/2}]^2 dx \leqq \frac{1}{2\gamma_0} \prod_{k=1}^{n} \left[\frac{\gamma_k - \gamma_0}{\gamma_k + \gamma_0}\right]^2.$$

Let

$$A(x) = \frac{x^{\gamma_0+1/2}}{\gamma_0 + 1/2} + \frac{u_1 x^{\gamma_1+1/2}}{\gamma_1 + 1/2} + \cdots + \frac{u_n x^{\gamma_n+1/2}}{\gamma_n + 1/2},$$

$$B(x) = \frac{(\gamma_0 + 1/2)A(x)}{x^{1/2}} = x^{\gamma_0} + c_1 x^{\gamma_1} + \cdots + c_n x^{\gamma_n},$$

$$C(x) = x^{\gamma_0-1/2} + u_1 x^{\gamma_1-1/2} + \cdots + u_n x^{\gamma_n-1/2}.$$

Then

$$A(x) = \int_0^x C(x)dx,$$

and

$$[A(x)]^2 = \left[\int_0^x C(x)dx\right]^2 \leqq \int_0^x dx \cdot \int_0^x [C(x)]^2 dx$$

$$= x \cdot \int_0^x [C(x)]^2 dx \leqq x \int_0^1 [C(x)]^2 dx$$

$$\leqq \frac{x}{2\gamma_0} \prod_{k=1}^{n} \left[\frac{\gamma_k - \gamma_0}{\gamma_k + \gamma_0}\right]^2 \qquad \text{on } [0, 1].$$

Therefore on $[0, 1]$

$$|A(x)| \leqq \frac{x^{1/2}}{(2\gamma_0)^{1/2}} \prod_{k=1}^{n} \frac{|\gamma_k - \gamma_0|}{\gamma_k + \gamma_0},$$

$$|B(x)| \leqq \frac{2\gamma_0 + 1}{2(2\gamma_0)^{1/2}} \prod_{k=1}^{n} \frac{|\gamma_k - \gamma_0|}{\gamma_k + \gamma_0}.$$

If $\gamma_0 = \gamma_i$ for some i, then the u's, and hence the c's, may obviously be chosen so that $|B(x)| \equiv 0$, and the inequality still holds.

It may now be shown that if $\gamma_0 = rH$, $(i-1)H < \gamma_i \leqq iH$, then

$$L_{r,n}(x) = |B(x)| \leqq \frac{2rH + 1}{2(2rH)^{1/2}} \cdot \frac{1}{C_r(2r-1) \cdot C_{2r}(n+r)} \qquad \text{on } [0, 1],$$

for $1 \leqq r \leqq n$. The proof depends upon the fact that for $\gamma_0 \geqq \gamma_k$,

$$\frac{\gamma_0 - \gamma_k}{\gamma_0 + \gamma_k} \leqq \frac{rH - (h-1)H}{rH + (h-1)H} = \frac{r - h + 1}{r + h - 1},$$

with a similar situation for $\gamma_0 \leqq \gamma_k$. If, in particular $\gamma_i = iH$ for some i, $\gamma_0 = iH$, then $L_{r,n}(x) \equiv 0$, and the inequality is still true.

3. **Degree of approximation.** This theorem will now be proved.

THEOREM I. *Consider the sequence* $\mu_0, \mu_1, \mu_2, \cdots$, *for which* $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, *where* $\mu_{n+1} - \mu_n \leqq H$ *for all* n, *and* $\lim_{n\to\infty} \mu_n = \infty$. *If* $f(x)$ *satisfies a Lipschitz condition* H, *constant* λ, *on* $[0, 1]$, *then for every integer* $n > 0$ *there exists* $P_{\mu_n}(x)$, *a polynomial in* x^μ *of degree* μ_n, *such that on* $[0, 1]$ $|f(x) - P_{\mu_n}(x)| \leqq K\lambda/\mu_n$, *where* K *depends on* H *only*.

To prove this theorem, a polynomial $P_{mH}(x) = a_0^{(m)} + a_1^{(m)} x^H + \cdots + a_{2m-2}^{(m)} x^{(2m-2)H}$ is constructed so that on $[0, 1]$ $|f(x) - P_{mH}(x)| \leqq D\lambda/m$, with bounds on the a's as given in the last section. Unless $rH = \gamma_k$, each x^{rH} is then approximated by a polynomial $P^{(r)}_{\gamma_{4m-4}}(x) = -c_1 x^{\gamma_1} - \cdots - c_{4m-4} x^{\gamma_{4m-4}}$, where the set of γ's is a subsequence of the set of μ's, $\gamma_0 = \mu_0$, γ_i is the smallest μ_n such that $(i-1)H < \gamma_i \leqq iH$, $i \geqq 1$; and $|x^{rH} - P^{(r)}_{\gamma_{4m-4}}(x)| = L_{r,4m-4}(x)$. Let

$$P_{\gamma_{4m-4}}(x) = a_0^{(m)} + \sum_{h=1}^{2m-2} a_h^{(m)} P^{(h)}_{\gamma_{4m-4}}(x).$$

Then

$$|f(x) - P_{\gamma_{4m-4}}(x)| \leqq \frac{D\lambda}{m}, \qquad \text{if } m = 1,$$

$$\leqq \frac{D\lambda}{m} + \sum_{k=1}^{m-1} P_k + \sum_{k=1}^{m-1} Q_k, \quad \text{if } m = 2, 3, \cdots,$$

where P_k and Q_k are the respective bounds of $|a_{2k}^{(m)}| \cdot |L_{2k,4m-4}(x)|$ and $|a_{2k-1}^{(m)}| \cdot |L_{2k-1,4m-4}(x)|$, as previously obtained. On calculation it is found that

$$\frac{P_{k+1}}{P_k} = \frac{4(k+1)H + 1}{4kH + 1} \cdot \frac{k^{1/2}}{(k+1)^{1/2}} \cdot \frac{2k+1}{4m + 2k - 3} \cdot \frac{m + k}{2m + k - 1} \cdot \frac{2k}{2m - k - 2} \cdot \frac{m - k - 1}{4m - 2k - 5},$$

$$\frac{Q_{k+1}}{Q_k} = \frac{2(2k+1)H + 1}{2(2k-1)H + 1} \cdot \frac{(2k-1)^{1/2}}{(2k+1)^{1/2}} \cdot \frac{2k}{4m + 2k - 3} \cdot \frac{m + k - 1}{2m + k - 2} \cdot \frac{2k - 1}{2m - k - 2} \cdot \frac{m - k - 1}{4m - 2k - 3},$$

for $k = 1, 2, \cdots, m-2$; $m = 3, 4, 5, \cdots$. If now each fraction is re-

placed by its maximum for the given range of k with m fixed, and then by its maximum for the given range of m, it follows that, since $(8H+1)/(4H+1) \leqq 2$, $(6H+1)/(2H+1) \leqq 3$, $P_{k+1}/P_k \leqq 2/9$, $Q_{k+1}/Q_k \leqq 1/3$; hence $P_k \leqq (2/9)^{k-1}P_1$, $Q_k \leqq (1/3)^{k-1}Q_1$. Finally

$$|f(x) - P_{\gamma_{4m-4}}(x)| \leqq \frac{D\lambda}{m} + \frac{9}{7}P_1 + \frac{3}{2}Q_1, \qquad m = 1, 2, \cdots.$$

Now direct calculation shows that

$$P_1 \leqq \frac{(4H+1)\pi\lambda}{45H^{1/2}m}, \qquad Q_1 \leqq \frac{(2H+1)\pi\lambda}{5(2H)^{1/2}m}, \qquad m = 2, 3, \cdots,$$

the equalities holding for $m=2$. Hence it appears that

$$|f(x) - P_{\gamma_{4m-4}}(x)| \leqq \frac{L\lambda}{m}, \qquad m = 1, 2, 3, \cdots,$$

where L depends on H only.

Now let n be an arbitrary integer, and let $m=(n+4)/4$, $(n+3)/4$, $(n+2)/4$, $(n+1)/4$, respectively, according as n has the form $4k$, $4k-1$, $4k-2$, $4k-3$. In any event $4m-4 \leqq n < 4m$, $\gamma_n \leqq nH$. Let the corresponding $P_{\gamma_{4m-4}}(x)$ be denoted by $P_{\gamma_n}(x)$, a polynomial of degree γ_n. Since $1/m \leqq 4/n \leqq 4H/\gamma_n$ it follows that on $[0, 1]$

$$|f(x) - P_{\gamma_n}(x)| \leqq \frac{4HL\lambda}{\gamma_n}.$$

Consider the original sequence x^{μ_0}, x^{μ_1}, x^{μ_2}, $\cdots$. Let $P_{\mu_n}(x)$ be a polynomial of degree μ_n formed from this sequence, $P_{\mu_n}(x) = P_{\gamma_k}(x)$ where γ_k is the largest of the γ's in $\mu_0, \mu_1, \cdots, \mu_n$. Now $\mu_n \leqq kH$, since otherwise γ_k would not be the largest γ in the given set of μ's. If $k \geqq 2$, then $2\gamma_k > (2k-2)H \geqq kH \geqq \mu_n$, $1/\gamma_k < 2/\mu_n$, and $|f(x) - P_{\mu_n}(x)| \leqq 8HL\lambda/\mu_n$. If $k=0$, 1 then $P_{\mu_n}(x) = P_{\gamma_0}(x)$, $\mu_n \leqq H$, $1 \leqq H/\mu_n$, and $|f(x) - P_{\mu_n}(x)| \leqq L\lambda/1 \leqq HL\lambda/\mu_n$. Hence the first of these two inequalities on $f(x)$ serves for all n, and the theorem is established.

This result may be extended to $[0, b]$, $b>0$, by assuming that $f(x)$ satisfies a Lipschitz condition H, constant λ, on $[0, b]$ and substituting $x=by$. Then there exists $\overline{P}_{\mu_n}(y)$ of degree μ_n such that on $[0, 1]$

$$|f(by) - \overline{P}_{\mu_n}(y)| \leqq \frac{Kb^H\lambda}{\mu_n},$$

whence on $[0, b]$

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{Kb^H\lambda}{\mu_n},$$

with $P_{\mu_n}(x) = \overline{P}_{\mu_n}(y)$.

The result may also be extended to $[a, b]$, where $a>0$, by defining $f(x)\equiv f(a)$ on $[0, a]$. In this case the condition on $f(x)$ may be replaced by one of the two simpler conditions.[6]

THEOREM II. *Suppose $f(x)$ is continuous on $[0, 1]$ and let*[6] $\omega_H(\delta)=\max|f(x_2)-f(x_1)|$ *for all x_1, x_2 on $[0, 1]$ satisfying the condition $|x_2^H-x_1^H|\leqq\delta$, and $\omega_H(\delta)\equiv\omega_H(1)$ if $\delta>1$. Then for each $\mu_n>0$ there exists a polynomial $P_{\mu_n}(x)$ such that on $[0, 1]$*

$$|f(x)-P_{\mu_n}(x)|\leqq\overline{K}\omega_H\left(\frac{1}{\mu_n}\right)$$

with $\overline{K}=K+2$.

Let

$$\phi(x)=a_ix^H+b_i,\qquad \left(\frac{i}{\mu_n}\right)^{1/H}\leqq x\leqq\left(\frac{i+1}{\mu_n}\right)^{1/H},\quad i=0, 1, \cdots,$$

$$a_i=\mu_n\left\{f_1\left[\left(\frac{i+1}{\mu_n}\right)^{1/H}\right]-f_1\left[\left(\frac{i}{\mu_n}\right)^{1/H}\right]\right\},$$

$$b_i=(i+1)f_1\left[\left(\frac{i}{\mu_n}\right)^{1/H}\right]-if_1\left[\left(\frac{i+1}{\mu_n}\right)^{1/H}\right],$$

where $f_1(x)\equiv f(x)$, $[0, 1]$, $f_1(x)\equiv f(1)$, $x\geqq 1$. Then

$$|\phi(x_2)-\phi(x_1)|\leqq\mu_n\omega_H\left(\frac{1}{\mu_n}\right)|x_2^H-x_1^H|,\qquad [0, 1],$$

$$|f(x)-\phi(x)|\leqq 2\omega_H\left(\frac{1}{\mu_n}\right),\qquad [0, 1].$$

It follows from the first of these two inequalities that a $P_{\mu_n}(x)$ exists such that $|\phi(x)-P_{\mu_n}(x)|\leqq K\omega_H(1/\mu_n)$ on $[0, 1]$, and the theorem follows. This theorem may also be extended to $[a, b]$, $0\leqq a\leqq b$, in which case

$$|f(x)-P_{\mu_n}(x)|\leqq K'\omega_H\left(\frac{b^H-a^H}{\mu_n}\right).$$

THEOREM III. *Let μ_n be given as before and let $\mu_\alpha=H$ be the smallest of the μ's for which $\mu_{n+1}-\mu_n\leqq\mu_\alpha$ for all n. Suppose $f(x)$ has a derivative for every x on $[a, b]$, and that $x^{1-H}f'(x)$ is continuous on $[a, b]$. Let $\eta_r=\mu_{r+\alpha}-H$ for all $r\geqq 0$. Suppose that there is given a sequence of positive numbers $\{\epsilon_n\}$, and suppose that for each ϵ_n there exists $P_n(x)$, a*

[6] Hereafter the function $\omega_H(\delta)$ will be called a "modulus of continuity H."

polynomial in x^n of degree $\eta_{n-\alpha}$, such that on $[a, b]$ $|x^{1-H}f'(x)-P_n(x)| \leqq \epsilon_n$. Then, for every $n>\alpha$ there exists $P_{\mu_n}(x)$, a polynomial in x^n of degree μ_n, such that on $[a, b]$

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{K''\epsilon_n}{\mu_n}$$

where K'' depends on H, a, and b.

Let

$$R_n(x) = \int_a^x x^{H-1}P_n(x)dx,$$

and let $r_n(x)=f(x)-R_n(x)$. Then, with the possible exception of $x=a$, $r_n'(x)$ may be found; hence $x^{1-H}r_n'(x)$, from which it follows that $|r_n'(x)| \leqq x^{H-1}\epsilon_n$. Since

$$r_n(x_2) - r_n(x_1) = \int_{x_1}^{x_2} r_n'(x)dx,$$

it follows that on $[a, b]$

$$|r_n(x_2) - r_n(x_1)| \leqq \frac{\epsilon_n}{H}|x_2^H - x_1^H|.$$

Therefore there exists $\overline{R}_n(x)$ in x^n of degree μ_n which approximates $r_n(x)$ with error at most $K\epsilon_n/H\mu_n$ and, with $P_{\mu_n}(x)=R_n(x)+\overline{R}_n(x)$,

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{K\epsilon_n}{H\mu_n}.$$

From Theorem III it follows that if $x^{1-H}f'(x)$ satisfies a Lipschitz condition H, constant λ, on $[a, b]$, then for every $n>\alpha$ there exists $P_{\mu_n}(x)$ such that

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{K_1\lambda}{\mu_n(\mu_n - H)}.$$

If $x^{1-H}f'(x)$ has a modulus of continuity H on $[a, b]$, then for every $n>\alpha$ there exists a polynomial $P_{\mu_n}(x)$ of degree μ_n for which

$$|f(x) - P_{\mu_n}(x)| \leqq \frac{K_2}{\mu_n}\omega_H\left(\frac{b^H - a^H}{\mu_n - H}\right).$$

4. **Uniform convergence.** With the sequence $\{x^{\mu_i}\}$ and a non-negative function $\rho(x)$, integrable on $[a, b]$ with positive integral, poly-

nomials $p_{\mu_n}(x)$ in x^μ of degree μ_n can be constructed for every μ_n by well known methods so that these polynomials will be orthonormal with respect to $\rho(x)$ on $[a, b]$. If $f(x)$ is an arbitrary continuous function, it follows as a formal result

$$f(x) \sim \sum_{n=0}^{\infty} a_n p_{\mu_n}(x), \tag{1}$$

where

$$a_n = \int_a^b f(t) p_{\mu_n}(t) \rho(t) dt.$$

In the following the results of the previous section are combined with certain results from the theory of orthogonal polynomials to give theorems on convergence for the above series expansion.[9] It will be necessary at this point to suppose that the μ's are integers, since Bernstein's theorem is implicitly involved in the following. It is to be observed that this restriction admits sequences so irregular that the results obtained are by no means trivial, and further that the admission of any set of commensurable exponents would not be a material generalization.

In the following statement of those results from orthogonal polynomial theory[10] which are applied to the present problem of convergence, all polynomials are polynomials in x^μ, of degree given by their subscript, and the use of ϵ_n implies that there exists a polynomial $P_{\mu_n}(x)$ such that on $[a, b]$, $|f(x) - P_{\mu_n}(x)| \leqq \epsilon_n$. If $\overline{Q}_{\mu_n}(x)$ is a polynomial minimizing[11]

$$\int_a^b \rho(x)[f(x) - Q_{\mu_n}(x)]^2 dx,$$

if $\rho(x)$ has a positive lower bound on $[a, b]$, then there are constants C_1, C_2, independent of x and n, such that

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[9] For the sequence $1, x^{\mu_1}, \cdots, x^{\mu_{N-1}}; x^\mu, x^{\mu+\mu_1}, \cdots, x^{\mu+\mu_{N-1}}; \cdots; x^{m\mu}, x^{m\mu+\mu_1}, \cdots, x^{m\mu+\mu_{N-1}}, \cdots$, where $0 < \mu_1 < \mu_2 < \cdots < \mu_{N-1} < \mu$, a recursion formula and hence a Christoffel-Darboux formula can be given for the orthonormal sequence. It is then possible to construct a theory for point-by-point convergence in the usual way. Since, on the one hand, there are no non-trivial results on boundedness of $p_{\mu_n}(x)$; and, on the other hand, no very interesting extensions of Bernstein's and Markoff's theorems are available, the results which are obtainable are omitted.

[10] D. Jackson, *Certain problems of closest approximation*, this Bulletin, vol. 39 (1933), pp. 903–906, Theorems 11, 15.

[11] It is a well known fact that $\overline{Q}_{\mu_n}(x) = \sum_{r=0}^{n} a_r p_{\mu_r}(x)$. Statements (A) and (B) therefore refer to the error of approximation by partial sums of the series for $f(x)$.

(A) $$|f(x) - \overline{Q}_{\mu_n}(x)| \leqq C_1 \mu_n \epsilon_n, \qquad a \leqq x \leqq b,$$

(B) $$|f(x) - \overline{Q}_{\mu_n}(x)| \leqq C_2 \mu_n^{1/2} \epsilon_n, \qquad c \leqq x \leqq d,$$

where $a < c < d < b$.

The problem of convergence then reduces to one of placing sufficient conditions on $f(x)$ so that $\mu_n \epsilon_n \to 0$ or $\mu_n^{1/2} \epsilon_n \to 0$. It is assumed in the following theorems that $\rho(x)$ is bounded from zero and integrable. From the results of the preceding sections values of ϵ_n are obtained which lead to the following theorems.

THEOREM IV. *If $f(x)$ has a derivative for which $x^{1-H}f'(x)$ satisfies a Lipschitz condition H on $[a, b]$, then the series* (1) *converges uniformly to $f(x)$ throughout $[a, b]$, and the magnitude of the remainder after n terms does not exceed $O[1/(\mu_n - H)]$.*

THEOREM V. *If $f(x)$ has a derivative for which $x^{1-H}f'(x)$ is continuous on $[a, b]$ and has a modulus of continuity H, $\omega_H(\delta)$,* (1) *converges uniformly to $f(x)$ throughout $[a, b]$, and the magnitude of the remainder after n terms does not exceed $O[\omega_H((b^H - a^H)/(\mu_n - H))]$.*

THEOREM VI. *If $f(x)$ satisfies a Lipschitz condition H on $[a, b]$,* (1) *converges uniformly to $f(x)$ throughout $[c, d]$, where $a < c < d < b$, and the magnitude of the remainder after n terms does not exceed $O(1/\mu_n^{1/2})$.*

THEOREM VII. *If $f(x)$ has a modulus of continuity H, $\omega_H(\delta)$, for which $\lim_{\delta=0} \omega_H(\delta)/\delta^{1/2} = 0$,* (1) *converges uniformly to $f(x)$ on $[c, d]$, where c and d have the same meaning as before.*

UNIVERSITY OF IDAHO

GENERALIZED FISCHER GROUPS AND ALGEBRAS[1]

RICHARD H. BRUCK

Introduction. In this paper we shall be concerned with the structure of the rational representation of certain sets of matrices, to which we give the name *generalized Fischer sets*. If K is any field, ϕ any fixed automorphism of K, and A any matrix with elements in K, we use the notation A^ϕ for the ϕ-automorph of A; that is, the matrix obtained from A by subjecting each of its elements to the automorphism ϕ. Again, we denote by A^* the transposed ϕ-automorph A'^ϕ of A.

DEFINITION. *A set $\mathfrak{M}$ of n-rowed square matrices which contains A^* if it contains A is defined to be a generalized Fischer set.*

Generalized Fischer groups, semi-groups and algebras are similarly defined.

In either of the special cases (1) $K=k$, a real field, ϕ is the identity automorphism; (2) $K=k^+=k(-1)^{1/2}$, ϕ is the operation of taking the conjugate complex, the set $\mathfrak{M}$ will be called a *Fischer set*. Fischer groups were probably named[2] by M. Schiffer, who, in 1933, proved in an unpublished work that every such group is completely reducible. This result has also been given by Specht [3], and will again be derived for all Kronecker product representations in the present paper (Theorem I, §4). In §1 we give a partial converse in the cases of the field of all reals (Example (6)), and the field of all complex numbers (Example (5)); this is summed up in Theorem II (§4).

Unlike Fischer sets, generalized Fischer sets and their rational representations are not always completely reducible; the regular representation of a finite group over a field of prime characteristic dividing the order of the group is a case in point (§1, Example (8)). When ϕ is non-involutary, the most we can give concerning the structure of g.F. sets is contained in Lemma II (§4) and Lemma IV (§5). But when ϕ is an involutary automorphism, a more satisfactory result is

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[1] The following is essentially contained in the author's doctorate thesis [1], written under the direction of Professor Richard Brauer. Professor Brauer has also offered many helpful suggestions in connection with the present paper. The thesis undertook a general study of $GL(n)$, and employed the results for specific calculation of the irreducible characters of $GL(4)$ over an infinite modular field.

[2] They were considered earlier by E. Fischer [2], who proved that the rational integral invariants of a Fischer group possessed a finite integrity basis.

given by Theorem III (§6); this states that each Kronecker product representation[3] of a g.F. set may be written in the form

$$\begin{pmatrix} \mathfrak{A} & & \\ & \mathfrak{B} & \\ . & & \mathfrak{C} \end{pmatrix}$$

where $\mathfrak{B}$ is in completely reduced form and $\mathfrak{C}$ is dual to $\mathfrak{A}$ in a sense which we will not describe at this point.

The main tool of the paper is the scalar product of two forms (or vectors) of the linear vector space S upon which the transformations of $\mathfrak{S}=\pi_f(\mathfrak{M})$ are performed. This is introduced in §2.

1. **Examples of generalized Fischer sets.** In this section proofs are omitted. The first two examples and the last hold true if the word set is replaced by group, semi-group or algebra. We use without specific mention material from [4], [5] and [6].

(1) *The direct sum of a finite number of g.F. sets (w.r.t. the same automorphism ϕ) is a g.F. set.*

(2) *The direct product of two g.F. sets (w.r.t. the same automorphism ϕ) is a g.F. set.*

(3) *A general linear group is a g.F. group w.r.t. every automorphsim ϕ of the underlying field.*

(4) *A total matric algebra is a g.F. algebra w.r.t. every ϕ.*

(5) *A semi-simple algebra of matrices over an algebraically closed field is similar to a g.F. algebra w.r.t. every ϕ.*

(6) *A semi-simple algebra of matrices over the field of all real numbers is similar to a Fischer algebra.*

(7) *Any group of matrices leaving invariant the bilinear form*

$$x^*y = \sum_{i=1}^{n} x_i^{\phi} y_i$$

is a g.F. group.

(8) *The right-hand (or first) regular representation $\mathfrak{R}$ of a finite group G is a g.F. group, and the linear closure of $\mathfrak{R}$ is a g.F. algebra, w.r.t. any ϕ, even if the underlying field is modular.*

(9) *If $\mathfrak{M}$ is a g.F. set w.r.t. the automorphism ϕ, its commutator algebra $\mathfrak{C}$ is a g.F. algebra w.r.t. the inverse automorphism $\theta=\phi^{-1}$.*

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[3] By the fth Kronecker product representation of a set $\mathfrak{M}$ we mean simply the set $\mathfrak{S}=\pi_f(\mathfrak{M})$ consisting of the fth Kronecker product [4] of each matrix of $\mathfrak{M}$. If $\mathfrak{M}$ is a (semi-) group, $\mathfrak{S}$ forms a representation in the usual sense. The corresponding statement in the case of an algebra is true only if $f=1$; thus the paper gives information only about the algebra itself.

2. **Kronecker product representations.** The matrices of a g.F. set $\mathfrak{A}$ may be considered as transformations of a contravariant vector x^i

$$A: \quad x'^i = \sum_{j=1}^{n} a_1^{ij} x^j. \tag{1}$$

Then the fth Kronecker product $\mathfrak{S} = \pi_f(\mathfrak{A})$ may be regarded as a set of transformations of the linear vector space S of all forms

$$\begin{aligned} F(T) &= \sum c_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}, \\ G(T) &= \sum d_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}, \end{aligned} \tag{2}$$

where $T^{i_1 i_2 \cdots i_f}$ is an arbitrary contravariant tensor with f indices. If under the transformation (1), T becomes T', then $F(T)$ becomes a new form $F_{(A)}(T')$ defined by

$$F(T) = F_{(A)}(T'). \tag{3}$$

We define the form $F_{(A)}(T)$ (in T, not T'!) to be the *transform of F by A*. Since

$$F_{(I)} = F, \qquad F_{(A)(B)} = F_{(AB)}, \tag{4}$$

it follows that if $\mathfrak{A}$ is a (semi-) group the set of transformations under $\mathfrak{A}$ of a basis of forms of S is a representation of $\mathfrak{A}$ similar to $\pi_f(\mathfrak{A})$.

We define the *scalar product* of the two forms F, G of (2) by

$$F \circ G = \sum c^{\theta}_{i_1 i_2 \cdots i_f} d_{i_1 i_2 \cdots i_f}, \qquad \theta = \phi^{-1}. \tag{5}$$

Clearly $F \circ G$ is a nondegenerate bilinear form in the two n^f-dimensional vectors F^{θ}, G whose coordinates are, respectively, $c^{\theta}_{i_1 i_2 \cdots i_f}$ and $d_{i_1 i_2 \cdots i_f}$. It has all the properties of such a bilinear form; we single out

$$(cF) \circ G = c^{\theta}(F \circ G), \qquad F \circ (cG) = c(F \circ G), \tag{6}$$

where c is any element of the underlying field. Again, it is readily verified that

$$F_{(A^*)} \circ G = F \circ G_{(A)}, \qquad A^* = A'^{\phi}. \tag{7}$$

These remarks and equations are fundamental to the present paper. Equation (7) explains the necessity for restricting attention to g.F. sets. Although all subsequent results are stated for the above Kronecker product representation they may be applied to others as well;[4] for this reason there would be some value in adopting a more axiomatic treatment. The essentials are: a vector space S with a g.F.

[4] In [1] the results were applied to Kronecker products of power products of alternating representations (that is, those afforded by skew-symmetric tensors) of $GL(n)$.

set $\mathfrak{A}$ of operators, as indicated by (4), and a nondegenerate scalar product $F \circ G$ bilinear in F^θ and G with the additional property (7).

We shall wish to refer to the following special properties which do not hold generally:

$$(H): \quad G \circ F = (F \circ G)^\theta, \qquad \phi = \theta. \tag{8}$$

This is true when ϕ is an involutary automorphism, and means that $F \circ G$ is a "hermitian" form.

$$(F): \quad G \neq 0 \text{ implies } G \circ G \neq 0. \tag{9}$$

This is certainly true in each of the following cases: (1) $K = k$, a real field, ϕ is the identity automorphism; (2) $K = k^+$, ϕ is the operation of taking the conjugate complex. Thus (F), and indeed also (H), may be assumed in dealing with Fischer sets.

3. **Modules.** Any subspace $S_1 \subset S$ which is invariant under the g.F. set $\mathfrak{A}$ we will call a *module*. Let the module S_1 have basis F_i $(i = 1, 2, \cdots, \sigma_1)$, and define the transpose of the matrix M_1 by

$$M_1' = (F_i \circ F_j), \tag{10}$$

where i, j are, respectively, the row and column index of the σ_1-rowed square matrix on the right. M_1 will be referred to as *the matrix associated with the module* S_1. Its rank is invariant under change of basis of S_1. If M_1 is nonsingular, S_1 is a *nonsingular* module. If M_1 is the zero matrix, S_1 is a module *of rank zero*. Since S_1 is a module, $F_{i(A)}$ is in S_1:

$$F_{i(A)} = \sum_{j=1}^{\sigma_1} s_{ij}^{(1)}(A) \cdot F_j. \tag{11}$$

If we set

$$s_1(A) = (s_{ij}^{(1)}(A)) \tag{12}$$

then the set $\mathfrak{S}_1$ of matrices $s_1(A)$ forms a representation of $\mathfrak{A}$.

DEFINITION. *Let $\mathfrak{A}$ be mapped upon a set $\mathfrak{U}$ of rectangular matrices: $A \to u(A)$. Then $\overline{\mathfrak{U}}$ denotes the set of matrices $\bar{u}(A) = [u(A^*)]'^\theta$.*

Note that if $\phi = \theta$ is involutary, $\bar{u}(A) = u^*(A^*)$. Moreover if $\mathfrak{R}$ is a representation of a semi-group $\mathfrak{G}$, then $\overline{\mathfrak{R}}$ is also a representation.

LEMMA I. *With the above notations*

$$M_1 \cdot \overline{\mathfrak{S}}_1 = \mathfrak{S}_1 \cdot M_1. \tag{13}$$

PROOF. By (7)

$$F_{j(A^*)} \circ F_i = F_j \circ F_{i(A)},$$

and by (6)

$$\sum_k F_k \circ F_i \cdot s_{jk}^{(1)\theta}(A^*) = \sum_l s_{il}^{(1)}(A) \cdot F_j \circ F_l.$$

But in matrix notation, if i, j are taken as row and column index, respectively, the last equation is identical with (13).

LEMMA II. *An irreducible module is either nonsingular or of rank zero.*

PROOF. Let S_1, M_1, $\mathfrak{S}_1$ be, respectively, the irreducible module, its associated matrix and the corresponding (irreducible) set of matrices. From Schur's lemma [5] and equation (13), M_1 is either nonsingular or zero.

We restrict our attention to modules of the two types mentioned in Lemma II, without, however, requiring them to be irreducible. The full space S is itself a nonsingular module. In fact, since the bilinear form $F \circ G$ is nondegenerate, the equation $F \circ G = 0$ holding for a fixed G and all F of S implies $G = 0$; it follows readily that the matrix M associated with S is nonsingular.

4. **Nonsingular modules.** Let the module S_1 be a proper subspace of S, in the sense that not every form of S is in S_1.

LEMMA III. *If S_1 is a nonsingular module, then S is the direct sum of S_1 and another nonsingular module S_2, and $\mathfrak{S}$ is decomposable:*

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 \\ 0 & \mathfrak{S}_2 \end{pmatrix}. \tag{14}$$

PROOF. Let F_i $(i = 1, 2, \cdots, \sigma_1)$ be a basis of S_1, and let these with G_α $(\alpha = \sigma_1 + 1, \cdots, \sigma)$ form a basis of S. Let G be any form of S. Then since the matrix M associated with S is nonsingular, we may uniquely determine elements $u_1, u_2, \cdots, u_{\sigma_1}$ such that

$$F = u_1 F_1 + u_2 F_2 + \cdots + u_{\sigma_1} F_{\sigma_1}$$

satisfies $F_i \circ F = F_i \circ G$, all i. Thus the form $G' = G - F$ satisfies $F_i \circ G' = 0$, for all i. Therefore, without loss of generality we may assume that the equations

$$F_i \circ G_\alpha = 0 \tag{15}$$

hold for all i, α. The matrix M associated with S then takes the form

$$M = \begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix} \tag{16}$$

where $M_2 = (G_\alpha \circ G_\beta)'$ is, with M and M_1, nonsingular. Since S_1 is a module, $\mathfrak{S}$ and $\overline{\mathfrak{S}}$ are given by

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 \\ \mathfrak{U} & \mathfrak{S}_2 \end{pmatrix}, \qquad \overline{\mathfrak{S}} = \begin{pmatrix} \overline{\mathfrak{S}_1} & \overline{\mathfrak{U}} \\ 0 & \overline{\mathfrak{S}_2} \end{pmatrix}. \tag{17}$$

But Lemma I, quoted for S, takes the form $M\overline{\mathfrak{S}} = \mathfrak{S}M$. From (16) and (17) follows $0 = \mathfrak{U}M_1$, $\mathfrak{U} = 0$. Thus $\mathfrak{S}$ assumes the form (14), and the set of all linear combinations of the G_α is a module; nonsingular, since M_2 is nonsingular.

When property (F) holds (see equation (9)), S clearly possesses no modules of rank zero. Hence we have this lemma.

LEMMA IIIa. *If (F) is true in S, $\mathfrak{S}$ is completely reducible.*

As an immediate consequence we have the following theorem.

THEOREM I. *The Kronecker product representations of a Fischer set are completely reducible.*

In particular the theorem states that Fischer groups and algebras are completely reducible. As a partial converse we have from examples (6) and (5) of §1 that a semi-simple algebra over the field of all real numbers or the field of all complex numbers is similar to a Fischer algebra.

THEOREM II. *Let $\mathfrak{G}$ be a (semi-) group of matrices over k or $k(-1)^{1/2}$, (k the field of all real numbers). Let $\mathfrak{A}$ be the linear closure of $\mathfrak{G}$. Then a necessary and sufficient condition that $\mathfrak{G}$ (and its Kronecker product representations) be completely reducible is that $\mathfrak{A}$ be similar to a Fischer algebra.*

5. **Modules of rank zero.** We now prove this lemma.

LEMMA IV. *If S_1 is a module of rank zero, then, in the sense of similarity,*

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & & \\ & \mathfrak{S}_2 & \\ * & & \overline{\mathfrak{S}_1} \end{pmatrix}. \tag{18}$$

PROOF. Let F_i $(i = 1, 2, \cdots, \sigma_1)$, be a basis of S_1, and let G_α $(\alpha = \sigma_1 + 1, \cdots, \sigma)$ complete the basis of S. Consider the matrix

$$U = (F_i \circ G_\alpha) \tag{19}$$

and the form $F = v_1F_1 + v_2F_2 + \cdots + v_{\sigma_1}F_{\sigma_1}$. If the matrix U had rank less than σ_1, we could choose a set of v's, not all zero, so that $F \circ G_\alpha = 0$ for all α. This, coupled with the fact that $F \circ F_i = 0$ for all i (since S_1 is a module of rank zero), would imply $F \circ G = 0$ for every G of S, in contradiction to the fact that the bilinear form $F \circ G$ is nondegenerate. Thus U has rank σ_1, and there exist (cf. [7]), two nonsingular matrices $P = (p_{ij})$ and $Q = (q_{\alpha\beta})$ such that $PUQ = (0 \mid I_{\sigma_1})$. We make the transformation of basis given by

$$\overline{F}_i = \sum p^{\phi}_{ij} F_j, \qquad \overline{G}_\alpha = \sum q_{\beta\alpha} G_\beta \tag{20}$$

so that

$$\overline{U} = (\overline{F}_i \circ \overline{G}_\alpha) = PUQ = (0 \mid I_{\sigma_1}). \tag{21}$$

Without loss of generality we may assume that (21) holds for U. We rename the last basis elements, calling them H_i $(i = 1, 2, \cdots, \sigma_1)$, so that our basis is now

$$F_1, F_2, \cdots, F_{\sigma_1}, G_{\sigma_1+1}, \cdots, G_{\sigma-\sigma_1}, H_1, H_2, H_{\sigma_1},$$

where the G's now appear only when $\sigma - 2\sigma_1 > 0$, but the H's necessarily appear. Calculation of M for this basis gives, in view of (21),

$$M = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ I_{\sigma_1} & * & * \end{pmatrix}. \tag{22}$$

We may assume

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 & 0 \\ * & \mathfrak{S}_2 & \mathfrak{U} \\ * & * & \mathfrak{S}_3 \end{pmatrix}, \qquad \overline{\mathfrak{S}} = \begin{pmatrix} \overline{\mathfrak{S}}_1 & * & * \\ 0 & \overline{\mathfrak{S}}_2 & * \\ 0 & \overline{\mathfrak{U}} & \overline{\mathfrak{S}}_3 \end{pmatrix}. \tag{23}$$

Using Lemma I, or $M\overline{\mathfrak{S}} = \mathfrak{S}M$, we obtain: by comparing the blocks (2, 1), $\mathfrak{U} = 0$; by comparing the blocks (3, 1), $\mathfrak{S}_3 = \overline{\mathfrak{S}}_1$. This completes the proof.

Lemma IV seems to be the most we can say when ϕ is a non-involutary automorphism.

Assume that $\phi = \theta$ is involutary, so that property (H) (equation (8)) holds for the scalar product. In this case (22) may be replaced by

$$M = \begin{pmatrix} 0 & 0 & I_{\sigma_1} \\ 0 & N & * \\ I_{\sigma_1} & * & * \end{pmatrix}, \qquad N = (G_k \circ G_l)', \tag{22a}$$

where h and k have the range σ_1+1 to $\sigma-\sigma_1$. The set S_2 of all linear combinations of the G_h is a linear vector space. For S_2 we may obtain a useful scalar product of two forms

$$
\begin{aligned}
G &= x_{\sigma_1}G_{\sigma_1} + \cdots + x_{\sigma-\sigma_1}G_{\sigma-\sigma_1}, \\
R &= y_{\sigma_1}G_{\sigma_1} + \cdots + y_{\sigma-\sigma_1}G_{\sigma-\sigma_1},
\end{aligned} \tag{24}
$$

by merely taking over the scalar product for S

$$
G \circ R = \sum n_{hk}x_h^{\phi}y_k, \qquad n_{hk} = G_h \circ G_k. \tag{25}
$$

From (22a), $\det M = \pm \det N \neq 0$, hence $N=(n_{hk})$ is nonsingular, $G \circ R$ is nondegenerate. In order to obtain a property comparable to (7) we must redefine the transform of a form of S_2 by a matrix of $\mathfrak{A}$. Since, by Lemma IV, S_2 is invariant (modulo S_1) under $\mathfrak{A}$,

$$
G_{(A)} = G(A) + F(A) \tag{26}
$$

where $G(A)$, $F(A)$ are uniquely determined forms of S_2, S_1. We define $G(A)$ to be the transform (*in* S_2) of G by A. Then, by virtue of (22a),

$$
G(A^*) \circ R = G_{(A^*)} \circ R = G \circ R_{(A)} = G \circ R(A). \tag{27}
$$

Formula (27) gives the desired property. There results this lemma.

LEMMA IVa. *If the automorphism ϕ is involutary, the representation $\mathfrak{S}_2$ of Lemma* IV *may be associated with a vector space S_2, with operator-set $\mathfrak{A}$, which possesses a nondegenerate (hermitian) scalar product satisfying* (7). *Thus the preceding methods and results may be applied to S_2 and $\mathfrak{S}_2$ just as they were to S and $\mathfrak{S}$.*

6. **The structure of** S. Lemmas III and IV may be combined in a number of ways to give information concerning the structure of S and $\mathfrak{S}$. This information is annoyingly limited, however, in case we cannot assume the conclusion of Lemma IVa. The following theorem therefore concerns only g.F. sets defined relative to an involutary automorphism.

THEOREM III. *Let $\mathfrak{A}$ be a set of n-rowed square matrices which contains $A^* = A'^{\theta}$ with every A, where θ is a fixed involutary automorphism of the underlying field. If $\mathfrak{R} = \{R(A)\}$ is any representation of $\mathfrak{A}$, define*

$$
\overline{\mathfrak{R}} = \{\overline{R}(A)\},
$$

where

$$
\overline{R}(A) = [R(A^*)]^*.
$$

Then the fth Kronecker product representation $\pi_f(\mathfrak{A})$ of $\mathfrak{A}$ is given in reduced form by

$$(28)\quad \pi_f(\mathfrak{A}) = \begin{pmatrix} \mathfrak{B}_1 & & & \\ & \mathfrak{B}_2 & & \\ & & \ddots & \\ & * & & \mathfrak{B}_\kappa \\ \hline & * & & \\ & & \mathfrak{C}_1 & \\ & & & \mathfrak{C}_2 \\ & & 0 & \ddots \\ & & & \mathfrak{C}_\lambda \\ \hline * & & * & \\ & & & \mathfrak{B}_\kappa \\ & & & \ddots \\ & & & \overline{\mathfrak{B}}_2 \\ & & * & \overline{\mathfrak{B}}_1 \end{pmatrix},$$

where the $\mathfrak{B}$'s and $\mathfrak{C}$'s are irreducible representations, and each $*$ denotes elements not necessarily zero.

The wording of the theorem is especially adapted to the case that $\mathfrak{A}$ is a (semi-) group. If $\mathfrak{A}$ is an algebra, the last sentence should be replaced by "Then $\mathfrak{A}$ is given in reduced form by $\cdots$."

PROOF. First we apply Lemma IV a number of times, assuming in each case that the S_1 is an irreducible module. After a finite number of steps, say κ, we must reach a projection space which contains no modules of rank zero. On applying Lemma III a number of times to this projection space we obtain the completely reduced block of $\mathfrak{C}$'s.

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A NEW PROOF OF THE CYCLIC CONNECTIVITY THEOREM

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The cyclic connectivity theorem was first proved for the plane in 1927 by G. T. Whyburn [5]. The extension of this theorem to metric space afforded some difficulty and the first proof [1] was long and tedious and complicated with convergence difficulties. A second and simpler proof appeared in 1931 [6], but in this proof it is necessary that quite a number of properties of Peano spaces be proved in advance.

This note attempts to give a new proof in which convergence troubles are encountered at just one point (step (b)) and in which just three theorems about Peano space need be known in advance: (A) *Every component of an open set is open.* (B) *Open connected sets are arc-wise connected.* (C) *The space is arc-wise locally connected.* Actually just two properties need to be established before cyclic connectivity can be proved, for the third theorem (C) is a simple consequence of the first two.[1] Thus the cyclic connectivity theorem may be established at the very beginning of the theory of Peano spaces and is available for use in studying other properties.

CYCLIC CONNECTIVITY THEOREM. *If no single point of a locally compact, connected and locally connected metric space separates the space between the two given points, there is a simple closed curve containing the two points.*

Let p and q be the two points. There exists an arc α of the space S with end points p and q by (B). We shall say that an arc β *spans* the point v of α if β has only its end points on α and v lies between these end points. We shall say that a set of arcs C *spans* a subset K of α if each point of K is spanned by some arc of the set C.

If an arc β exists with end points r and q and such that $\alpha \cdot \beta = r + q$, then step (d) in the proof has been achieved. Hence we consider only the case where no such arc exists. This assumption is used in the proof of step (b).

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[1] Let G be the component of $S(p, \epsilon)$ containing the point p. By (A), G is open. Then for some δ, $S(p, \delta) \subset G$. By (B) G is arc-wise connected. Hence every point of $S(p, \delta)$ may be joined to p by an arc in G, and thus in $S(p, \epsilon)$, which proves arc-wise local connectivity.

(a) *If r is a point of $\alpha-p-q$, there is an arc β spanning r.*

As r does not cut S between p and q, by (A) p and q belong to the same component C of $S-r$. Then C contains an arc γ with end points p and q by (A) and (B). On γ in the order from p to q there is a first point y of the subarc rq of α. On γ in the order y to p there is a first point x of the subarc rp of α. Then the subarc xy of γ is the desired arc β.

For any point r, let $f(r)$ be the greatest lower bound of the diameters of β, for all arcs β spanning r.

(b) $f(r)\to 0$ *as* $r\to q$.

If this is false, there is an $\epsilon>0$ and a sequence of points of α converging to q so that no one of these points may be spanned by an arc of diameter less than ϵ. We may choose ϵ so small that $S(q, \epsilon)$ is compact. Let us select a spanning arc for each point of the sequence. Infinitely many of these arcs are distinct since we assumed in the beginning that no one of them has q as an end point. Hence there exists an infinite set of arcs $\beta_1, \beta_2, \beta_3, \cdots$ such that (i) $\alpha\cdot\beta_i=x_i+y_i$, the end points of β_i, and x_i precedes y_i on α, (ii) diam $\beta_i\geqq\epsilon$, (iii) β_i spans a point p_i of α which cannot be spanned by any arc of diameter less than ϵ, (iv) we have the order $pp_1y_1p_2y_2\cdots q$ on α, (v) $y_i\to q$ and $\rho(y_i, q)<\epsilon/6$. On β_i in the order y_i to x_i let z_i be the first point such that $\rho(z_i, q)=\epsilon/3$. The set of points $\{z_i\}$ has a limit point z and we may assume $z_i\to z$. By (C) there exist arcs z_iz of diameter less than $\epsilon/6$ for i sufficiently large. Then $y_iz_i+z_iz+z_{i+1}z+y_{i+1}z_{i+1}$ contains an arc of diameter less than ϵ spanning p_{i+1}, which is a contradiction.

(c) *If r is any point of $\alpha-p-q$, there exists a countable sequence of arcs $\{\beta_i\}$ spanning the set $rq-q$ and such that* diam $\beta_i\to 0$.

Let $p_i\to q$ with the order $prp_1p_2\cdots q$. Let $p_0=r$ and α_i be the subarc $p_{i-1}p_i$ of α. From (a) there exists a family of arcs spanning α_i. From (b) these may be chosen of arbitrarily small diameter for i large. Using the Borel theorem there is a finite subfamily spanning α_i. The set of all these finite subfamilies is a countable set with the desired properties.

(d) *There is a simple closed curve containing q.*

Let u_i and v_i be the end points of β_i and suppose the order pu_iv_iq on α. From the method in which the arcs β_i were chosen we see that only a finite number span any one point and no one intersects more than a finite number of others. Of the finite set spanning r, choose the one whose end point v_i is nearest q on α in the order p to q. We may assume this is β_1. Of the finite set, if nonvacuous, of arcs β_i which intersect $\beta_1-u_1-v_1$, choose the one whose end point v_i follows all others on α. We may assume this is β_2. Of the finite sets, if nonvacu-

ous, which intersect $\beta_2 - u_2 - v_2$, we choose the one whose end point v_i follows all others on α, and we may suppose this is β_3. This process must terminate at some finite step for otherwise $\sum \beta_i$ would contain an arc having u_1 and q as end points and containing no other point of α.[2] This arc together with the subarc $u_1 q$ of α would give the desired simple closed curve.

If the process terminates at β_k, then $\sum_1^k \beta_i$ contains an arc γ_1 with end points $s_1 = u_1$ and $t_1 = v_k$ and containing no other point of α. We now define an arc γ_2 with end points s_2 and t_2 using exactly the process by which γ_1 was defined but starting with the arcs spanning v_k instead of r. We continue and define $\gamma_3, \gamma_4, \cdots$. The arcs γ_i are mutually exclusive except that t_i may coincide with s_{i+2}. On α we have the order $s_1 < s_2 < t_1 \leqq s_3 < t_2 \leqq s_4 < \cdots < q$. Then the desired two arcs forming a simple closed curve containing q are defined

$$\eta_1 = q + \sum \gamma_{2i-1} + \sum \text{subarcs } t_{2i-1} s_{2i+1} \text{ of } \alpha,$$

$$\eta_2 = q + \text{subarc } s_1 s_2 \text{ of } \alpha + \sum \gamma_{2i} + \sum \text{subarcs } t_{2i} s_{2i+2} \text{ of } \alpha.$$

From step (d) the cyclic connectivity theorem follows easily. We have a simple closed curve containing q. Similarly one contains p. If these simple closed curves have two points in common, their sum contains a simple closed curve containing p and q. If they have but one point in common, their sum plus an arc pq not containing this point will contain the desired simple closed curve. If they do not intersect, then their sum plus an arc joining them plus its finite spanning system enables one to choose the desired two arcs. In this case the selection of the two arcs follows the methods used in picking η_1 and η_2 but the whole process here is finite.

Remarks. The cyclic connectivity theorem is a special case of the n-Bogensatz where $n = 2$ and the two closed sets are single points. All proofs that have been given for the n-Bogensatz are extremely long and intricate [2, 3, 4, 7], and a simple proof of this important theorem would be a real contribution. Unfortunately the method used in this note does not appear to generalize to the higher values of n.

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[2] This arc would be given by q plus the subarc of β_1 from u_1 to the first point of β_2 on β_1, plus the subarc of β_2 from this point to the first point point of β_3, and so on. From diam $\beta_i \to 0$ it follows that this set is an arc with end points u_1 and q.

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INVERSES AND ZERO-DIVISORS

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It may happen that an element in a ring is both a zero-divisor and an inverse, that it possesses a right-inverse though no left-inverse, and that it is neither a zero-divisor nor an inverse. Thus there arises the problem of finding conditions assuring the absence of these paradoxical phenomena; and it is the object of the present note to show that chain conditions on the ideals serve this purpose. At the same time we obtain criteria for the existence of unit-elements.

The following notations shall be used throughout. The element e in the ring R is a *left-unit for the element* u in R, if $eu=u$; and e is a *left-unit for* R, if it is a left-unit for every element in R. Right-units are defined in a like manner; and an element is a *universal unit for* R, if it is both a right- and a left-unit for R.

The element u is a *right-zero-divisor*, if there exists an element $v\neq 0$ in R such that $vu=0$; and u is a *right-inverse in* R, if there exists an element w in R such that wu is a left-unit for u and a right-unit for R. Left-zero-divisors and left-inverses are defined in a like manner. Note that 0 is a zero-divisor, since we assume that the ring R is different from 0.

$L(u)$ denotes the set of all the elements x in R which satisfy $xu=0$; clearly $L(u)$ is a left-ideal in the ring R and every left-ideal of the form $L(u)$ shall be termed a *zero-dividing left-ideal*. *Principal left-ideals*[1] are the ideals of the form Rv for v in R and the ideals vR are the principal right-ideals.

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[1] This is a slight change from the customary terminology.

LEMMA 1. *If vu is a left-unit for u and a right-unit for R, then $uv=vu$, $Rv=R$ and $L(u)=0$ are equivalent properties.*

PROOF. If $uv=vu$, then $R=Rvu=Ruv\leqq Rv\leqq R$, or $R=Rv$. If $R=Rv$, and if $xu=0$, then there exists an element y in R such that $x=yv$; and $L(u)=0$ is a consequence of $y=yvu=xu=0$. If $L(u)=0$ then $(vu-uv)u=vuu-uvu=u-u=0$ implies that $vu-uv$ is in $L(u)$ and that therefore $vu=uv$.

EXAMPLE 1. Let G be the additive group of all the countably infinite sequences of numbers from the commutative field F, considered as an abelian operator group over F; and denote by R the ring of all the F-admissible automorphisms of G (that is, of all the linear transformations of G over F). Elements u, v, w in R are defined by

$$(a_1, \cdots)^u = (0, a_1, \cdots), \qquad (a_1, \cdots)^v = (a_1, 0, \cdots),$$
$$(a_1, \cdots)^w = (a_2, a_3, \cdots)$$

and they satisfy $uv=0$, $uw=1$, $wu\neq 1$. Thus a left-inverse may be a left-zero-divisor.

LEMMA 2. *Each of the following properties of the ring R implies all the others.*

(i) *There exists a universal unit in R.*

(ii) *There exists a right-unit for R, and $x=0$ is a consequence of $Rx=0$.*

(iii) *There exists one and only one right-unit for R.*

(iv) *There exist a right-unit for R and a left-unit for R.*

PROOF. (ii) is an obvious consequence of (i). If (ii) holds, and if u and v are right-units for R, then $x(u-v)=xu-xv=0$ for every element x in R; thus $u=v$ and (iii) is an implication of (ii). If (iii) holds, if e is the uniquely determined right-unit for R, and if x and y are elements in R, then $x(e+y-ey)=xe+xy-xey=x+xy-xy=x$ for every x in R. Thus $e+y-ey$ is a right-unit for R; and the equality of all right-units implies that $y=ey$, that is, that e is also a left-unit for R; and (iv) is therefore a consequence of (iii). If (iv) holds, then there exist a right-unit u and a left-unit v for R; and (i) may be inferred from $u=vu=v$.

The importance of condition (ii) may be seen from the fact that the elements x in R which satisfy $Rx=0$ form a two-sided ideal T whose square is 0.

LEMMA 3. *If the minimum condition is satisfied by the principal left-ideals in the ring R, if J is a principal left-ideal in R and if $Jv=R$ for v an element in R, then $J=R$.*

REMARK. Omitting the word "principal" throughout we obtain a true statement that may be proved in a similar fashion.

PROOF. If $R=Jv$ for J a principal left-ideal and v an element in R, then there exists a smallest principal left-ideal $S=Rs$ such that $R=St$ for some t in R. Hence $s=rst$ for r in R. Since $Rrs\leqq Rs=S$, and since $(Rrs)t^2=R(rst)t=Rst=St=R$, it follows from our choice of S that $Rrs=S$; and hence $S=Rs=Rrst=St=R$, a fact which proves our contention.

THEOREM 1. *The element z in the ring R with minimum condition for principal left-ideals is a right-inverse in R if, and only if, z is not a right-zero-divisor in R.*

PROOF. Assume first the existence of an element v in R such that vz is a left-unit for z and a right-unit for R. Then $J=Rv$ is a principal left-ideal and it follows from Lemma 3 and the equation $Jz=Rvz=R$ that $R=J=Rv$; and hence it follows from Lemma 1 that z is not a right-zero-divisor.

If there exist elements in R which are not right-zero-divisors, then among these elements there is one, say v, with minimal (principal left-ideal) Rv. Since $xv^2=0$ implies $xv=0$, and since this implies $x=0$, it follows that v^2 is not a right-zero-divisor. Thus $Rv^2\leqq Rv$ and the minimum property of Rv imply $Rv^2=Rv$. Hence for every element r in R there exists an element r' in R such that $rv=r'v^2$ or $(r-r'v)v=0$. Since v is not a right-zero-divisor, $r-r'v=0$ or $r=r'v$. Thus we have shown that $R=Rv$; and this fact makes it evident that $R=Rz$ for every z which is not a right-zero-divisor in R.

If the element z in R is not a right-zero-divisor in R, then $R=Rz$; and there exists therefore one (and only one) element e in R such that $ez=z$. Since $(r-re)z=rz-rz=0$ for every r in R, it follows that e is both a left-unit for z and a right-unit for R. Since the existence of an element w satisfying $wz=e$ is a consequence of $R=Rz$, it has been shown that z is a right-inverse in R.

An immediate consequence of this theorem is the following fact.

COROLLARY. *The ring R with minimum condition for principal left-ideals contains a right-unit for R if (and only if) at least one element in R is not a right-zero-divisor.*

The impossibility of substituting the maximum condition for the minimum condition in Theorem 1 and its corollary may be seen from the example of the even rational integers. The example of the ring of all the matrices

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

with coefficients from some commutative field shows that the conditions of the corollary are not sufficient for the existence of a universal unit.

THEOREM 2. *If there exists a universal unit* 1 *in the ring R, if the minimum condition is satisfied by both the principal right- and the principal left-ideals in R, then each of the following properties of the element z in R implies all the others*:

(a) *z is not a right-zero-divisor.*
(b) *z is a right-inverse.*
(c) *z is not a left-zero-divisor.*
(d) *z is a left-inverse.*

That (b) implies (c) and that (d) implies (a) are obvious consequences of Lemma 1; and that (a) implies (b) and that (c) implies (d) may be inferred from Theorem 1 and Lemma 2.

The impossibility of substituting maximum conditions for the minimum conditions in this theorem may be seen from the example of the ring of all the rational integers.

THEOREM 3. *If the minimum condition is satisfied by the principal left-ideals and by the zero-dividing left-ideals in the ring R, then* $R=Rz$ *is a necessary and sufficient condition for z to be a right-inverse in R.*

PROOF. If z is a right-inverse in R, then vz is for some v in R a right-unit for R and $R=Rvz\leqq Rz\leqq R$ or $R=Rz$.

If, conversely, $R=Rz$, then $z=ez$ for some e in R. The principal left-ideal $J=Re$ satisfies $Jz=Rez=Rz=R$. Hence it follows from Lemma 3 that $R=Re$ for every left-unit e of the element z.

Among the left-units for z there exists one, say f, with minimal $L(f)$. Then it is a consequence of $R=Rf$ that $f=gf$ for some g in R. Since $gz=g(fz)=(gf)z=fz=z$, and since $xg=0$ implies $0=xgf=xf$, it follows that g is a left-unit for z and that $L(g)\leqq L(f)$. Hence $L(g)=L(f)$ is a consequence of our choice of f. Since $(g^2-g)f=f-f=0$, the element g^2-g is in $L(f)=L(g)$ so that $(g^2-g)g=0$ or $g^3=g^2$. Consequently the element $e=g^2$ satisfies $e^2=g^4=g^3=g^2=e$ and $ez=g^2z=z$, that is, the element e is an idempotent and a left-unit for z. Thus e is an idempotent satisfying $R=Re$ (as has been shown in the second paragraph of the proof); and the left-unit e for z is therefore a right-unit for R. Finally there exists an element w in R such that $wz=e$, since $R=Rz$; and this completes the proof of the fact that z is a right-inverse in R.

The following fact is an immediate consequence of Theorem 3.

COROLLARY. *The ring R with minimum condition for principal and for zero-dividing left-ideals contains a right-unit for R if, and only if, R is a principal left-ideal in R.*

As an application of the preceding results we prove the following statement.

The ring R is a (not necessarily commutative) field if, and only if,

(a) *0 is the only right-zero-divisor in R,*

(b) *the minimum condition is satisfied by the principal left-ideals in R.*

It is obvious that the two conditions are necessary and that neither of them could be omitted. If the conditions (a) and (b) are satisfied by R, then every element $z \neq 0$ in R is a right-inverse by Theorem 1. Since R contains therefore a right-unit for R, it follows from Lemma 2 that R contains a universal unit 1; and it follows from Lemma 1 that every element $z \neq 0$ in R is both a right- and a left-inverse. Hence R is a field.

LEMMA 4.[2] *If the maximum condition is satisfied by the zero-dividing left-ideals in the ring R, if the left-ideal J and the element z in R satisfy $Jz = R$, then $L(z) = 0$ and $J = R$.*

PROOF. If $R = Jz$ for some left-ideal J and some element z in R, then among these elements z there exists one, say w, with maximal $L(w)$. If W is some left-ideal such that $R = Ww$, then $R = Ww \leqq Rw = Ww^2 \leqq R$ or $R = Ww^2$. Hence it follows from our choice of w and the obvious inequality $L(w) \leqq L(w^2)$ that $L(w) = L(w^2)$. Every element in $L(w)$ has the form $x = yw$ for y in W, since $R = Ww$. If yw is in $L(w)$, then y is in $L(w^2) = L(w)$, $yw = 0$ and consequently $L(w) = 0$. Thus we have shown that $L(z) = 0$ whenever $R = Jz$ for some left-ideal J in R.

If $Jz = R$, then for every element r in R there exists an element j in the left-ideal J such that $jz = rz$. This implies $j = r$, since z is not a right-zero-divisor. This completes the proof of the fact that $J = R$ and $L(z) = 0$ are consequences of $Jz = R$.

THEOREM 4. *Suppose that the maximum condition is satisfied by the zero-dividing left-ideals in the ring R.*

(a) *uv is a right-unit for R if, and only if, vu is a right-unit for R.*

(b) *The right-unit vu for R is a left-unit for u if, and only if, $uv = vu$.*

(c) *R contains a right-unit (for $\mathfrak{R}$) if, and only if, R is a principal left-ideal (in R).*

PROOF. If vu is a right-unit for R, then $R = Rvu$; and it follows from

[2] Cf. Lemma 3.

Lemma 4 that $L(u)=0$, $Rv=R$, $L(v)=0$, $Ru=R$. If r is any element in R, then $(r-ruv)u=ru-ru(vu)=ru-ru=0$. Since $L(u)=0$, we find that $r=ruv$ for every r in R. This proves (a); and (b) may be obtained as an immediate consequence of Lemma 1. If $R=Rz$ for some z in R, then it follows from Lemma 4 that $L(z)=0$. Furthermore, there exists an element e in $R=Rz$ such that $z=ez$. If r is any element in R, then $(r-re)z=rz-rz=0$. The element e is a right-unit for R, since $L(z)=0$ implies $r=re$ for every r. This completes the proof of the theorem.

THEOREM 5. *If there exists a universal unit* 1 *in* R, *if the minimum condition for principal left-ideals or the maximum condition for zero-dividing left-ideals is satisfied in* R, *then the relations* $uv=1$ *and* $vu=1$ *imply each other.*

PROOF. If the maximum condition is satisfied by the zero-dividing left-ideals in R, then our contention is an immediate consequence of Theorem 4, (a) and Lemma 2. If the minimum condition is satisfied by the principal left-ideals in R, then $uv=1$ implies $L(u)=0$. By Theorem 1 there exists an element w such that $wu=1$. But $w=wuv=v$ so that $vu=wu=1$.

The impossibility of omitting the chain conditions in Theorem 5 is a consequence of Example 1.

The following special case of Theorem 5 may be worth mentioning, since it solves a problem of importance in the theory of matrices.[3] If R is a ring with universal unit 1, then denote by R_n the ring of all square matrices with n rows and n columns whose coefficients are in R. If the maximum (minimum) condition is satisfied by the left-ideals in R, then the same condition holds in R_n; and hence it is a consequence of Theorem 5 that the matrices A and B in R_n satisfy either $AB=1$ *and* $BA=1$ or neither of these relations; in particular it is impossible that a matrix in R_n is both a zero-divisor and an inverse in R_n.

A right- or left-ideal J in the ring R is termed nilpotent, if $J^i=0$ for some positive integer i. The sum $P=P(R)$ of all the nilpotent right- and left-ideals in R is a two-sided ideal in R which is called the *radical* of R. Hopkins[4] has shown that the radical itself is nilpotent, if the minimum condition is satisfied by the left-ideals in R.

THEOREM 6. *There exists a left-unit*[5] *for the ring* R *with minimum condition for left-ideals if, and only if,*

[3] Cf., for example, B. L. van der Waerden, *Moderne Algebra*, vol. 2, Berlin, 1927, pp. 114–115.

[4] C. Hopkins, Annals of Mathematics, (2), vol. 40 (1939), pp. 712–730; Theorem 1.4, p. 714.

[5] Note that the preceding criteria assured the existence of right-units for the ring R.

(i) $Rx=0$ *implies* $x=0$,

(ii) $P^iRx=xP=0$ *implies* $P^ix=0$ *for* $0<i<m$ (*where* m *is the smallest positive integer such that* $P^m=0$).

PROOF. If R contains a left-unit for every element then x is always an element of Rx; and this shows the necessity of conditions (i) and (ii).

Suppose that conditions (i) and (ii) are satisfied by the ring R. From the choice of the integer m it follows that $0=P^m<P^{m-1}$ (using the notation $P^0=R$). If x is an element not 0 in P^{m-1}, then $Px=0$ so that $P<R$ is a consequence of condition (i). Hence[6] there exists in R an idempotent $e\neq 0$ such that $R=Re+P$.

Denote by T the set of all the elements x in R which satisfy: $xP=0$; and denote by $S(i)$ (for positive i) the set of all those elements in T which satisfy: $P^ix=0$. Clearly T and therefore every $S(i)$ is a two-sided ideal in R. We are going to prove by complete induction that $eS(i)=S(i)$. This is true for $i=1$, since it follows from $Rx=Rex+Px$ and condition (i) that $x=0$ is the only element x in $S(1)$ which satisfies $ex=0$. Suppose now that $S(i)=eS(i)$ and that the element t in $S(i+1)$ satisfies $et=0$. Then $Rt=Ret+Pt=Pt\leqq PS(i+1)\leqq S(i)$, $P^iRt\leqq P^iS(i)=0$; and it follows from condition (ii) that $P^it=0$. The element t is therefore an element in $S(i)=eS(i)$, that is, $t=et=0$ or $S(i+1)=eS(i+1)$; and this completes the induction. In particular it follows that $T=eT$, since $T=S(m)$ is a consequence of $P^m=0$.

Denote by $T(i)$ the set of all the elements x in R which satisfy $xP^i=0$. Every $T(i)$ is a two-sided ideal in R. We are going to prove by complete induction that $T(i)=eT(i)$. This is true for $i=1$, since $T=T(1)$. Suppose that $T(i)=eT(i)$ and that the element v in $T(i+1)$ satisfies $ev=0$. Then $vP=0$, since $vP\leqq T(i)=eT(i)$ and $ev=0$. Thus v is an element in $T=eT$ and $v=ev=0$. This shows that $T(i+1)=eT(i+1)$; and this completes the induction. Since $P^m=0$ implies $T(m)=R$, we have proved in particular that $R=eR$; and the idempotent e is therefore a left-unit for R.

We note that we proved the following fact. If the conditions (i) and (ii) of Theorem 6 are satisfied by the ring R, if the radical of R is nilpotent, and if the idempotent e in R satisfies $R=Re+P$, then e is a left-unit for R; and we mention that the minimum condition for zero-dividing left-ideals would be sufficient to assure the existence of such an idempotent e.

One verifies readily that the conditions (i) and (ii) of Theorem 6 are consequences of each of the following (not necessary) conditions

[6] Cf. Hopkins, loc. cit., Theorems 4.1, 4.2, p. 721.

(a) $P=0$; (b) condition (i), and $xP=0$ implies $Px=0$; (c) condition (i), and $RP^i \leqq P^iR$.

That condition (ii) is indispensable for the validity of Theorem 6 may be seen from the following example. Denote by R the ring of all the matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & e \end{pmatrix}$$

with coefficients from some (commutative) field F. It is readily verified that $Rx=0$ implies $x=0$; that $yR=0$ implies $y=0$; and that this ring R does not contain a universal unit 1. We note that R would be a finite ring, if F were a finite field.

COROLLARY. *There exists a left-unit for the ring R with minimum condition for left-ideals if (and only if) there exists a left-unit for every element in R.*

PROOF. The existence of a left-unit for the element x in R is equivalent to the fact that x is an element in the left-ideal Rx. This shows that the conditions (i), (ii) of Theorem 6 are consequences of the condition of the corollary.

THEOREM 7. *There exists a universal unit in the ring R with minimum condition for left-ideals if, and only if,*

(1) $Rx=0$ *implies* $x=0$,

(2) $yR=0$ *implies* $y=0$,

(3) $RP=PR$.

PROOF. The necessity of the conditions is obvious. If the conditions are satisfied by R, then it follows from (1), (3) that conditions (i), (ii) of Theorem 6 are satisfied. Hence there exists a left-unit e for R; and it follows from Lemma 2 and condition (2) that e is a universal unit for R.

COROLLARY. *There exists a universal unit in the commutative ring R with minimum condition for ideals if, and only if, $Rx=0$ implies $x=0$.*

This is an immediate consequence of Theorem 7.

The impossibility of substituting the maximum condition for the minimum condition in Theorems 6 or 7 may be seen from the example of the even rational integers.

Comparing the results of this investigation one sees that minimum conditions are more powerful than maximum conditions. This phe-

nomenon is closely connected with Hopkins' theorem:[7] *If the ring R contains a left-unit or a right-unit for R, then the maximum condition for left-ideals in R is a consequence of the minimum condition for left-ideals in R.* We could not make use of this theorem, since we usually considered rings with minimum (maximum) condition for principal or zero-dividing left-ideals. On the other hand it is possible to improve slightly the theorem of Hopkins by applying our criteria for the existence of units.

It is well known that *the commutative law of addition* is a consequence of the customary postulates for a field, but not of the analogous postulates for a ring. To investigate this situation we consider a nonvacuous set R of elements which are connected by two operations: addition $u+v$ and multiplication uv, subject to the following rules:

I. R is a group under addition.

II. The product uv of the elements u and v in R is a uniquely determined element in R.

III. $u(v+w)=uv+uw$, $(u+v)w=uw+vw$.

We denote by R^2 the subgroup of the additive group R which is generated by all the products uv and by $C=C(R)$ the commutator subgroup of the additive group R. Both R^2 and C are two-sided ideals in R.

The addition is commutative in R^2, and $CR=RC=0$.

PROOF. If a, b, c, d are elements in R, then it follows from the distributive laws that

$$ad + ab + cd + cb = (a + c)(d + b) = ad + cd + ab + cb;$$

and hence it follows from the cancellation law of addition that

$$ab + cd = cd + ab.$$

Our contention is an immediate consequence of this equality.

It is now obvious that each of the following two conditions is sufficient to assure commutativity of addition.

(a) $R=R^2$.

(b) 0 is the only element in R which satisfies both $Rx=0$ and $xR=0$.

We note that each of these two conditions is necessary, but not sufficient for the existence of a universal unit in R.

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[7] Hopkins, loc. cit., Theorem 6.7, p. 728.